REPORT 1153

ON THE APPLICATION OF TRANSONIC SIMILARITY RULES TO WINGS OF FINITE SPAN [1]

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SUMMARY

The transonic aerodynamic characteristics of wings of finite span are discussed from the point of view of a unified small disturbance theory for subsonic, transonic, and supersonic flows about thin wings. Critical examination is made of the merits of the various statements of the equations for transonic flow that have been proposed in the recent literature. It is found that one of the less widely used of these possesses considerable advantages, not only from the point of view of a priori theoretical considerations but also of actual comparison of theoretical and experimental results. The similarity rules and known solutions of transonic-flow theory are reviewed, and the asymptotic behavior of the lift, drag, and pitching-moment characteristics of wings of large and small aspect ratio is discussed. It is shown that certain methods of data presentation are superior for the effective display of these characteristics.

INTRODUCTION

The small perturbation potential theory of transonic flow proposed apparently independently by Oswatitsch and Wieghardt, Busemann and Guderley, von Kármán (refs. 1 through 6), and others is now supplying a fund of information regarding transonic flow about aerodynamic shapes. Solutions have been given for two-dimensional flow around airfoils at both subsonic and supersonic speeds in papers by Guderley and Yoshihara, Vincenti and Wagoner, Cole, Trilling, Oswatitsch, Gullstrand (refs. 7 through 16), and others. In the application of these results to specific examples, two items of theoretical interest have been noted (see, in particular, refs. 8, 17, and 18): (a) The theoretical results appear to be applicable at Mach numbers far removed from 1 even though, in most cases, the results have been obtained from equations valid only in the immediate neighborhood of sonic speed. (b) In the application of theoretical results to specific examples at Mach numbers other than 1, it has been noted that certain ambiguities exist in the theoretical determination of aerodynamic quantities. It is one of the purposes of this report to investigate these two points. This is accomplished by examining transonic flow from the point of view of equations that are valid throughout the Mach number range rather than only in the neighborhood of sonic speed. Such an approach emphasizes the relation between the roles of linear theory and of nonlinear theory in the transonic range.

The similarity rules provided by the theory (refs. 5, 6, and 19 through 22) have also proved to be useful in the cor-relation and interpretation of experimental data. It is with the latter aspect of the transonic-flow problem that the present paper is primarily concerned. In this paper, the similarity rules and their application to the specific problem of concise presentation of lift, drag, and pitching-moment characteristics of wings are given in detail. The known solutions of two-dimensional transonic flow are reviewed and the asymptotic behavior of the aerodynamic characteristics of wings of large and small aspect ratios is examined. It is shown that certain methods of data presentation are advantageous for displaying these characteristics.

SYMBOLS

A	aspect ratio
$\tilde{A}$	$[U_o k(t/c)]^{1/3} A$
a	speed of sound
a_o	speed of sound in the free stream
a^*	critical speed of sound
b	wing semispan
C_D	drag coefficient
$\tilde{C}_D$	$\dfrac{(U_o k)^{1/3}}{(t/c)^{5/3}} C_D$
C_{D_o}	drag coefficient of symmetrical nonlifting wings
$\tilde{C}_{D_o}$	$\dfrac{(U_o k)^{1/3}}{(t/c)^{5/3}} C_{D_o}$
ΔC_D	contribution to drag coefficient due to lift
$(\widetilde{\Delta C_D/\alpha^2})$	$[U_o k(t/c)]^{1/3} (\Delta C_D/\alpha^2)$
C_L	lift coefficient
$(\widetilde{C_L/\alpha})$	$[U_o k(t/c)]^{1/3} (C_L/\alpha)$
C_m	pitching-moment coefficient
$(\widetilde{C_m/\alpha})$	$[U_o k(t/c)]^{1/3} (C_m/\alpha)$
C_p	pressure coefficient
$\tilde{C}_p$	$\dfrac{(U_o k)^{1/3}}{(t/c)^{2/3}} C_p$
$C_{p_{cr}}$	critical pressure coefficient
$C. P.$	center-of-pressure function
c	wing chord
c_{d_o}	section drag coefficient of symmetrical nonlifting airfoils
$\tilde{c}_{d_o}$	$\dfrac{(U_o k)^{1/3}}{(t/c)^{5/3}} c_{d_o}$
Δc_d	contribution to section drag coefficient due to lift
$(\widetilde{\Delta c_d/\alpha^2})$	$[U_o k(t/c)]^{1/3} (\Delta c_d/\alpha^2)$

c_l	section lift coefficient
$(\widetilde{c_l/\alpha})$	$[U_o k(t/c)]^{1/3} (c_l/\alpha)$
$\mathcal{D}$	drag function
$\mathcal{D}_c$	drag function for symmetrical nonlifting wings
$\mathcal{D}_\Delta$	drag due to lift function
d_o	section drag function for symmetrical nonlifting airfoils
d_Δ	section drag due to lift function
k	coefficient of nonlinear term of differential equation for φ. (See eqs. (7), (18), (19), (20), (23), and (35).)
$\mathcal{L}$	lift function
l	section lift function
$\mathcal{M}$	pitching-moment function
m_s	section pitching-moment function
M	local Mach number
M_o	free-stream Mach number
$\mathcal{P}$	pressure function
s	stretching factors defined in equation (B8)
t	maximum thickness of wing
U_o	free-stream velocity
$\overline{U},\overline{V}$	velocity components parallel and perpendicular, respectively, to the flow direction ahead of a shock
x,y,z	Cartesian coordinates where x extends in the direction of the free-stream velocity
$x_{c.p.}$	distance from wing leading edge to center of pressure
(Z/c)	ordinates of wing profiles in fractions of chord
α	angle of attack
$\tilde{\alpha}$	$\dfrac{\alpha}{(t/c)}$
γ	ratio of specific heats, for air $\gamma=1.4$
λ	arbitrary constant
ξ	$\dfrac{(M^2-1)}{[U_o k(t/c)]^{2/3}}$
ξ_o	$\dfrac{(M_o^2-1)}{[U_o k(t/c)]^{2/3}}$
τ	ordinate-amplitude parameter
Φ	velocity potential
φ	perturbation velocity potential

SUBSCRIPTS

l	values given by linear theory
W	conditions at the wing surface
1	conditions immediately upstream from shock
2	conditions immediately downstream from shock

FUNDAMENTAL CONCEPTS

BASIC EQUATIONS

The quasi-linear partial differential equation satisfied by the velocity potential Φ of steady isentropic flow of a perfect inviscid gas can be expressed in the form (ref. 23, p. 25)

$$(a^2-\Phi_x^2)\Phi_{xx}+(a^2-\Phi_y^2)\Phi_{yy}+(a^2-\Phi_z^2)\Phi_{zz}-2\Phi_x\Phi_y\Phi_{xy}-2\Phi_y\Phi_z\Phi_{yz}-2\Phi_z\Phi_x\Phi_{zz}=0 \qquad (1)$$

where the subscript notation is used to indicate differentiation and a is the local speed of sound given by the relation

$$a^2=a_o^2-\frac{\gamma-1}{2}(\Phi_x^2+\Phi_y^2+\Phi_z^2-U_o^2) \qquad (2)$$

In this latter equation U_o and a_o are, respectively, the velocity and speed of sound in the free stream and γ is the ratio of specific heats (for air $\gamma=1.4$).

Introducing the perturbation velocity potential φ, where

$$\varphi=-U_o x+\Phi \qquad (3)$$

it is possible to express equation (1) in terms of the derivatives of φ as follows:

$$(1-M_o^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=$$

$$\left\{ \begin{array}{l} \dfrac{\varphi_{xx}}{a_o^2}\left[(\gamma+1)U_o\varphi_x+\dfrac{\gamma+1}{2}\varphi_x^2+\dfrac{\gamma-1}{2}(\varphi_y^2+\varphi_z^2)\right]+ \\[2mm] \dfrac{\varphi_{yy}}{a_o^2}\left[(\gamma-1)U_o\varphi_x+\dfrac{\gamma-1}{2}(\varphi_x^2+\varphi_z^2)+\dfrac{\gamma+1}{2}\varphi_y^2\right]+ \\[2mm] \dfrac{\varphi_{zz}}{a_o^2}\left[(\gamma-1)U_o\varphi_x+\dfrac{\gamma-1}{2}(\varphi_x^2+\varphi_y^2)+\dfrac{\gamma+1}{2}\varphi_z^2\right]+ \\[2mm] 2\dfrac{\varphi_{xy}}{a_o^2}\varphi_y(U_o+\varphi_x)+ \\[2mm] 2\dfrac{\varphi_{xz}}{a_o^2}\varphi_z(U_o+\varphi_x)+ \\[2mm] 2\dfrac{\varphi_{yz}}{a_o^2}\varphi_y\varphi_z \end{array} \right\} \qquad (4)$$

If it is assumed that all perturbation velocities and perturbation velocity gradients (represented by first and second derivatives, respectively, of φ) are small and that only the first-order terms in small quantities need be retained, equation (4) simplifies to the well-known Prandtl-Glauert equation of linear theory

$$(1-M_o^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0 \qquad (5)$$

where the free-stream velocity is directed along the positive x axis as shown in figure 1 and where M_o is the Mach number of the free stream. It is well known that equation (5) leads
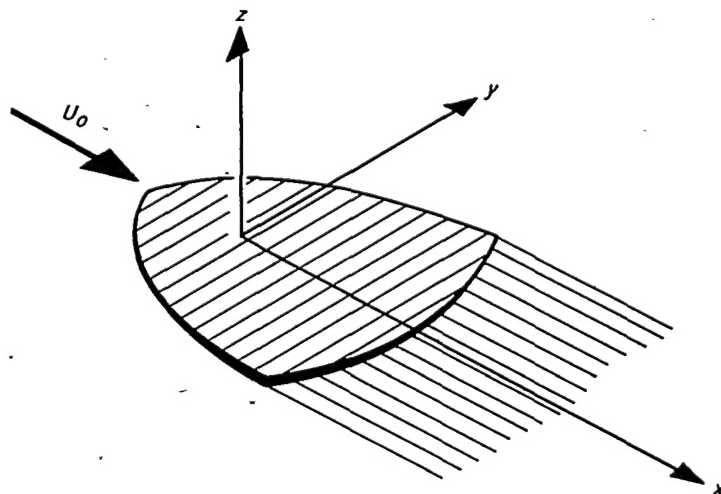

FIGURE 1.—View of wing and coordinate system.

to useful results in the study of subsonic and supersonic flows about thin wings and slender bodies but that it is incapable, in general, of treating transonic flows. The failure of linear theory in the transonic range is evidenced by the calculated value of φ_x growing to such magnitude that it can no longer be regarded as a small quantity when compared with U_o.

Second-order theory for thin wings would involve solution of the equation

$$(1-M_o{}^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=$$
$$M_o{}^2\left[\frac{\gamma+1}{U_o}\varphi_x\varphi_{xx}+\frac{\gamma-1}{U_o}\varphi_x(\varphi_{yy}+\varphi_{zz})+\frac{2}{U_o}(\varphi_y\varphi_{xy}+\varphi_z\varphi_{xz})\right] \quad (6)$$

Actually, we are interested in retaining higher-order terms only to the extent that is necessary to allow the study of transonic flow. Examination of the known characteristics of transonic flow fields indicates that the first term on the right can often become of importance and should be retained. The remainder of the terms on the right can never become large for transonic flows about thin wings at small angles of attack and can be safely disregarded. The simplified equation is

$$(1-M_o{}^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=M_o{}^2\frac{\gamma+1}{U_o}\varphi_x\varphi_{xx}=k\varphi_x\varphi_{xx} \quad (7)$$

It follows from the basic assumptions of small-disturbance transonic theory that equation (7) is valid everywhere in the flow field, both fore and aft of any shock waves, but cannot provide any information on the discontinuities in φ that represent the shock waves. This information must be obtained through consideration of the classical equations for the velocities on either side of an oblique shock. Thus if $\overline{U}_1$ represents the velocity immediately in front of the shock wave and $\overline{U}_2$ and $\overline{V}_2$ represent the velocity components parallel and perpendicular, respectively, to $\overline{U}_1$ that occur immediately behind the shock, the equation for the shock polar (ref. 23, p. 108) provides that

$$\overline{V}_2{}^2=(\overline{U}_1-\overline{U}_2)^2\frac{\overline{U}_1\overline{U}_2-a^{*2}}{\frac{2}{\gamma+1}\overline{U}_1{}^2-\overline{U}_1\overline{U}_2+a^{*2}} \quad (8)$$

where a^* represents the velocity of sound at a point where the local Mach number is unity and is commonly designated the critical speed of sound. It is related to the free-stream velocity and speed of sound by the following equation derived from equation (2):

$$\frac{\gamma+1}{2}a^{*2}=a_o{}^2+\frac{\gamma-1}{2}U_o{}^2 \quad (9)$$

Except for the important case of the bow wave in supersonic flow, $\overline{U}_1$ is not, in general, alined with the direction of the x axis, but is inclined a small angle. With the resolution into components parallel to the axes of the coordinate system and the retention of only the leading terms, in a manner similar to that used in the derivation of equation (7), equation (8) provides the following relations between the velocity components (potential gradients) immediately fore and aft of the shock:

$$(\varphi_{y_1}-\varphi_{y_2})^2+(\varphi_{z_1}-\varphi_{z_2})^2=(\varphi_{x_1}-\varphi_{x_2})^2\frac{[U_o{}^2-a^{*2}+U_o(\varphi_{x_1}+\varphi_{x_2})]}{\left(a^{*2}-U_o{}^2+\frac{2}{\gamma+1}U_o{}^2\right)} \quad (10)$$

This result can be put into a more desirable form by making use of the following relation derived directly from equation (9):

$$U_o{}^2-a^{*2}=\frac{2}{\gamma+1}\frac{M_o{}^2-1}{M_o{}^2}U_o{}^2 \quad (11)$$

Substitution of this expression into equation (10) and rearrangement of the terms yields the desired result

$$(1-M_o{}^2)(\varphi_{x_1}-\varphi_{x_2})^2+(\varphi_{y_1}-\varphi_{y_2})^2+(\varphi_{z_1}-\varphi_{z_2})^2=$$
$$M_o{}^2\frac{\gamma+1}{U_o}\left(\frac{\varphi_{x_1}+\varphi_{x_2}}{2}\right)(\varphi_{x_1}-\varphi_{x_2})^2=k\left(\frac{\varphi_{x_1}+\varphi_{x_2}}{2}\right)(\varphi_{x_1}-\varphi_{x_2})^2 \quad (12)$$

This equation corresponds to the shock-polar curve for weak shock waves inclined at any angle between that of normal shock waves and that of the Mach lines.

In addition to satisfying equations (7) and (12), the perturbation potential must provide flows compatible with the following physical requirements: (a) The flow must be uniform far ahead of the wind, and (b) the flow must be tangential to the wing surface. Therefore, the following boundary conditions are to be specified for the perturbation potential:

at $x=-\infty$

$$(\varphi_x)_o=(\varphi_y)_o=(\varphi_z)_o=0 \quad (13)$$

at the wing surface W

$$\left(\frac{\varphi_z}{U_o+\varphi_x}\right)_W=\frac{\partial Z}{\partial x} \quad (14)$$

where $\partial Z/\partial x$ refers to the local slope in the x direction of the wing surface. A systematic application of the perturbation analysis indicates that the boundary conditions specified on the wing surface should be simplified by approximating the fraction on the left by φ_z/U_o. Furthermore, it is consistent with the assumption of small disturbances to satisfy this boundary condition on the two sides of the xy plane rather than on the wing surface. Equation (14) is therefore replaced by

$$\left(\frac{\varphi_z}{U_o}\right)_{z=0}=\frac{\partial Z}{\partial x}=\tau\frac{\partial}{\partial(x/c)}f\left(\frac{x}{c},\frac{y}{b}\right) \quad (15)$$

where the shape of the wing profile is given by

$$Z/c=\tau f(x/c,y/b) \quad (16)$$

where $f(x/c,y/b)$ represents the ordinate-distribution function and τ is an ordinate-amplitude parameter. Note that, in general, a variation of τ represents a simultaneous change of the thickness ratio, camber, and angle of attack. In the special case of a nonlifting wing having symmetrical sections, τ is proportional to the thickness ratio; for lifting flat-plate wings of vanishing thickness, τ is proportional to the angle

of attack. In order to obtain unique and physically important solutions, it is necessary to assume the Kutta condition (that the flow leaves all subsonic trailing edges smoothly).

Upon solving the above boundary-value problem for the potential, one may determine the pressure coefficient by means of the formula

$$C_p = -\frac{2}{U_o} \varphi_x \qquad (17)$$

It should be noted that the results obtained by using the foregoing approximate equations might be expected to tend toward those of linear theory as the free-stream Mach number M_o departs far from unity. This follows from the fact that the product $\varphi_x\varphi_{xx}$ becomes small relative to the linear terms under this condition. Solutions of the equations for transonic flow found to date have all possessed this property.

COMPARISON WITH OTHER STATEMENTS OF THE TRANSONIC-FLOW EQUATIONS

As a result of minor variations in the perturbation analysis, recent papers have used at least four different relations for k, the coefficient of the nonlinear term in the simplified equation for the perturbation velocity potential. As indicated in the preceding paragraphs, straightforward development of the theory leads to the relation

$$k = M_o^2 \frac{\gamma+1}{U_o} \qquad (18)$$

This is sometimes simplified (e. g., refs. 22 and 24) to

$$k = \frac{\gamma+1}{U_o} \qquad (19)$$

by arguing that M_o can be set equal to unity in this term without much loss in accuracy, since the right-hand side of equation (7) is merely an approximation to allow the treatment of transonic flows and rapidly diminishes in magnitude as M_o departs from unity. In some treatments (e. g., refs. 16 and 21), equation (1) is divided by a^2, and the quotient $1/a^2$ in each term is expanded in a binomial series. When this is done, the coefficient k of the term involving $\varphi_x\varphi_{xx}$ is

$$k = M_o^2 \left[\frac{2+(\gamma-1)M_o^2}{U_o} \right] \qquad (20)$$

Still another expression for k is used by Oswatitsch in references 13 and 14. Two derivations are given, one based on mass-flow considerations (ref. 14) and the other (ref. 13) based on simplifying equation (1), under the assumption of nearly parallel flow, to

$$(1-M^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0 \qquad (21)$$

and expanding the variable coefficient $1-M^2$ in the series

$$1-M^2 = 1-M_o^2 - \frac{1-M_o^2}{a^*-U_o}\frac{\partial\varphi}{\partial x} + \cdots \qquad (22)$$

where M is the local Mach number and a^* is the critical speed of sound as defined in equation (9). Comparison of equations (21) and (22) with equation (7) shows the coefficient k in this approximation is

$$k = \frac{1-M_o^2}{a^*-U_o} \qquad (23)$$

It should be noted that the four alternative relations for k are identical for $M_o=1$ and all but that given by equation (19) are zero for $M_o=0$.

A similar situation arises in the derivation of the simplified equation for the shock polar. Here again the precise form of the expression for the coefficient k of equation (12) depends on the details of the perturbation analysis. The most important point from a practical point of view is that the same expression for k is used in both the equation for the potential and that for the shock polar, namely equations (7) and (12). While this point has not always been explicitly stated, it is actually a necessary condition for the existence of the transonic similarity rules.

Although each of the above alternative forms of the perturbation equations has been used at least once in the recent theoretical investigations of transonic flow, the most widely used set of equations are those derived under the more restrictive assumption that all velocities are small perturbations around the critical speed of sound a^* rather than around the free-stream velocity U_o. In the latter scheme, the perturbation potential is defined by (see, e. g., ref. 6 or 20)

$$\varphi' = -a^*x+\Phi \qquad (24)$$

and the resulting differential equation for φ' is

$$\varphi'_{yy}+\varphi'_{zz}=\frac{\gamma+1}{a^*}\varphi'_x\varphi'_{xx} \qquad (25)$$

The approximate relation for the shock polar is

$$(\varphi'_{y_1}-\varphi'_{y_2})^2+(\varphi'_{z_1}-\varphi'_{z_2})^2=\frac{\gamma+1}{a^*}\left(\frac{\varphi'_{x_1}+\varphi'_{x_2}}{2}\right)(\varphi'_{x_1}-\varphi'_{x_2})^2 \qquad (26)$$

The corresponding boundary conditions are specified as follows:

at $x = -\infty$

$$(\varphi'_x)_o = U_o-a^* \approx -\frac{a^*}{\gamma+1}(1-M_o^2), \qquad (\varphi'_y)_o=(\varphi'_z)_o=0$$

at the wing surface

$$(\varphi'_z)_{z=0} \approx a^*\tau\frac{\partial}{\partial(x/c)}f\left(\frac{x}{c},\frac{y}{b}\right) \qquad (27)$$

where the shape of the wing profile is still given by equation (16). The equation for the pressure coefficient is approximated similarly, thus,

$$C_p' \approx -\frac{2}{a^*}[\varphi'_x-(\varphi'_x)_o] \qquad (28)$$

This statement of the equations for transonic flow is clearly identical to those given previously, herein, when the free-stream Mach number is unity. Although the derivation of the a^* equations requires that the free-stream Mach number be very close to unity, these equations have been used with good success by a number of authors to calculate the aerodynamic forces on airfoils at Mach numbers considerably

removed from unity. In so doing, it has been suggested that it might be preferable to use more accurate relations for the pressure coefficient or the boundary conditions; for instance, it has been suggested that a^* be replaced with U_o in the equation for C_p'. This matter has been discussed at length in references 8, 17, and 18. Since no restriction requiring the Mach number to be near unity is made in the U_o analysis, it is informative to examine the relation between the results of the a^* and the U_o analyses. —This is done in Appendix A. It is found that the a^* analysis, if performed in a completely consistent manner using equations (24) through (28), yields values for C_p that are identical to those given by the more general U_o analysis using $k=(\gamma+1)/U_o$. This somewhat paradoxical result is achieved through the action of a number of compensating effects and only applies to the pressures and the forces and moments derivable therefrom. It should be noted, in particular, that the values of the local velocities and Mach numbers provided by the a^* analysis for flows having free-stream Mach numbers other than unity are in error. Throughout the remainder of this report, the discussion will be based on the U_o analysis.

A significant case where the alternative relations for k lead to different results is the prediction of the critical pressure coefficient $C_{p_{cr}}$, defined as the value of the pressure coefficient C_p at a point where the local Mach number is unity. It is important that a reasonably good approximation be maintained for the variation of $C_{p_{cr}}$ with M_o because shock waves make their first appearance, and the airfoil first experiences a pressure drag when C_p becomes more negative than $C_{p_{cr}}$ somewhere on the airfoil surface. In the present approximation, $C_{p_{cr}}$ corresponds to that value of C_p, and, hence, or φ_x, at which equation (7) changes locally from elliptic to hyperbolic type. This condition is recognized by the vanishing of the coefficient of φ_{xx}, thus

$$1-M_o^2-k(\varphi_x)_{cr}=0$$

or, in view of equation (17)

$$C_{p_{cr}}=-\frac{2}{U_o}(\varphi_x)_{cr}=-\frac{2}{kU_o}(1-M_o^2) \qquad (29)$$

The exact relation for isentropic flow is (e. g., ref. 25, p. 28)

$$C_{p_{cr}}=\frac{2}{\gamma M_o^2}\left[\left(\frac{2}{\gamma+1}+\frac{\gamma-1}{\gamma+1}M_o^2\right)^{\frac{\gamma}{\gamma-1}}-1\right] \qquad (30)$$

The variation of $C_{p_{cr}}$ with M_o has been computed by use of the exact relation and each of the four approximate relations. The results are presented graphically in figure 2. It may be seen that a reasonably good approximation for $C_{p_{cr}}$ is obtained over a wide Mach number range when k is taken as given in either equation (18) or (23), and that a somewhat greater error is incurred when equation (20) is used. On the other hand, equation (19) leads to a very poor approximation for $C_{p_{cr}}$.

Similar comparisons can be made for local Mach numbers M other than unity by noting that the coefficient $1-M_o^2-k\varphi_x$
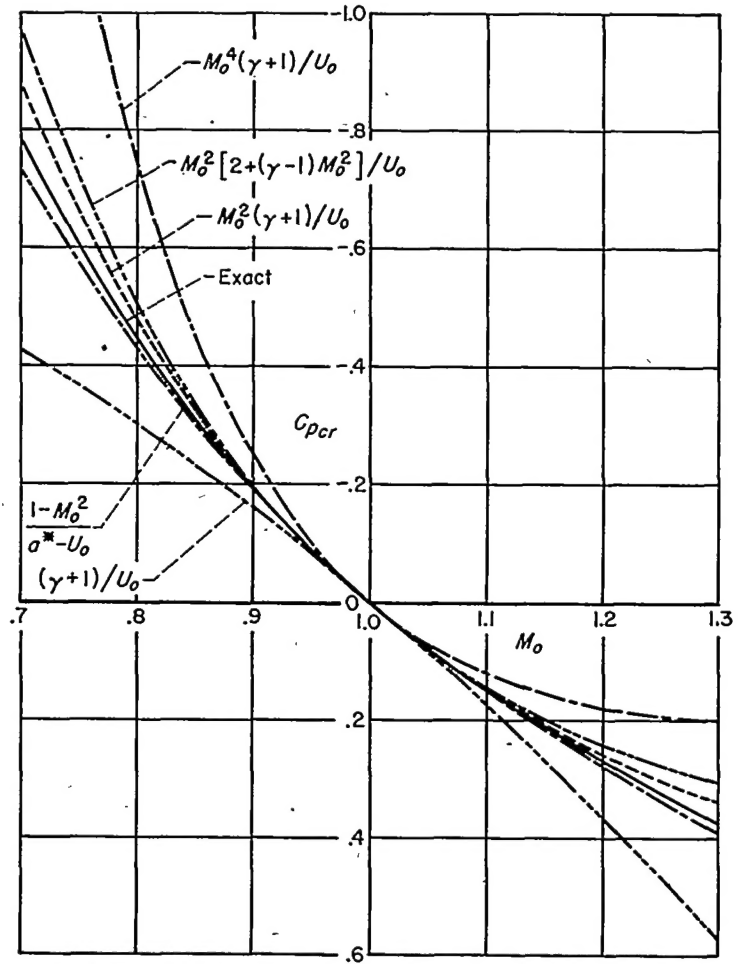

FIGURE 2.—Variation of critical pressure coefficient with Mach number.

of φ_{xx} in equation (7) corresponds, in the present approximation, to $1-M^2$, thus

$$1-M^2=1-M_o^2-k\varphi_x=1-M_o^2+\frac{kU_o}{2}C_p \qquad (31)$$

The corresponding exact relation for isentropic flow is

$$C_p=\frac{2}{\gamma M_o^2}\left[-1+\left(\frac{1+\frac{\gamma-1}{2}M_o^2}{1+\frac{\gamma-1}{2}M^2}\right)^{\frac{\gamma}{\gamma-1}}\right] \qquad (32)$$

The results so obtained are generally similar to those indicated in figure 2, although the relative accuracy of the better approximations changes somewhat with the situation. All the approximations are exact, of course, when $C_p=0$; on the other hand, none of the approximations are exact, except for isolated cases, when C_p is different from zero, even though all the approximations agree among themselves when the free-stream Mach number is unity. In order to provide some information regarding the errors that are likely to be incurred when C_p is not very small, figure 3 has been prepared illustrating the variation of local Mach number with pressure coefficient for a free-stream Mach number of unity.

A second case where the exact and approximate relations may be compared is furnished by considering the velocity
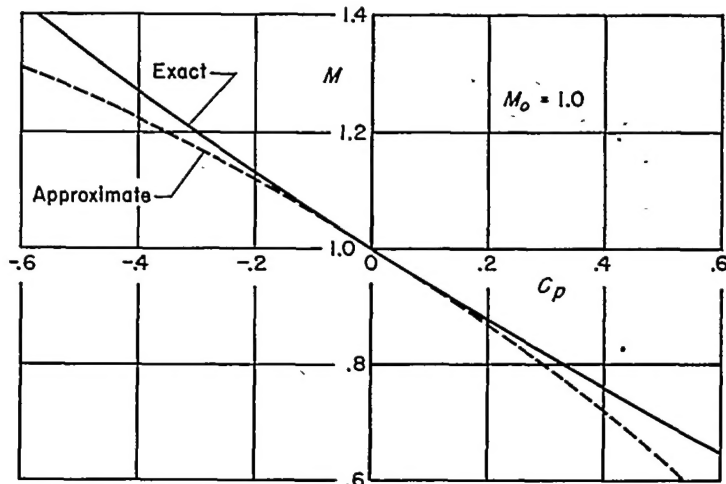
FIGURE 3.—Variation of local Mach number with pressure coefficient for $M_o=1$.

jump through a shock wave. If the flow ahead of the shock wave is uniform and parallel to the x axis, the results may be conveniently represented by the shock-polar diagram in which $\overline{V}_2/a^*$ is plotted as a function of $\overline{U}_2/a^*$. The exact relation is furnished by equation (8). The corresponding approximate relations are determined from equation (12) by setting φ_{x_1}, φ_{y_1}, φ_{y_2}, and φ_{z_1} equal to zero, whereby $U_o=U_1$, $M_o=M_1$, and

$$\varphi_{x_2}{}^2=\left[-(1-M_o{}^2)+\frac{k}{2}\varphi_{x_2}\right]\varphi_{x_2}{}^2 \tag{33}$$

Once the variation of φ_{x_2} with φ_{x_2} is determined for a given M_o, the corresponding variation of $\overline{V}_2$ with $\overline{U}_2$ may be readily determined since, for this case,

$$\overline{U}_2=U_o+\varphi_{x_2}, \qquad \overline{V}_2=\varphi_{x_2} \tag{34}$$

The variation of $\dfrac{\overline{V}_2}{a^*}$ with $\dfrac{\overline{U}_2}{a^*}$ for $M_o=1.2$ has been computed using both the exact and approximate relations, and the results are presented graphically in conventional shock-polar form in figure 4. This figure contains, in addition to curves for the four expressions for k discussed previously, a curve computed using

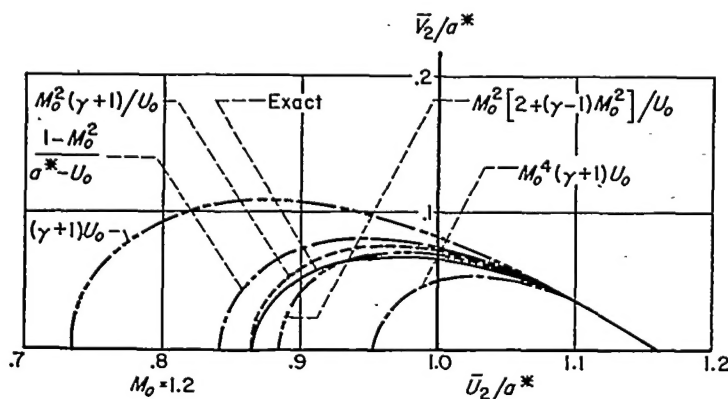$$k=M_o{}^4\frac{\gamma+1}{U_o} \tag{35}$$

FIGURE 4.—Exact and approximate shock polars.

This expression for k arises in a series expansion of the right-hand member of equation (10). (For sake of completeness, the corresponding variation of $C_{p_{cr}}$ with M_o is also included in figure 2.) Just as with the comparison of the critical pressure coefficient, the expressions for k given by equations (19) and (35) lead to poor approximations, whereas any other of the expressions lead to reasonably good approximations. A notable point is that the expression for k given by equation (18), that is $k=M_o{}^2(\gamma+1)/U_o$, leads to the exact relation for velocity jump through a normal shock wave.

In order to facilitate comparison with previous results and to achieve an economy of notation, the present analysis is carried as far as possible without specifying a particular relation for k. That is, the equations of the analysis and reduced parameters with which the results are expressed are written containing k which may be equated to any of the five stated relations, or in fact, to any coefficient that does not depend on x,y,z or φ. The actual values of the pressure coefficient and Mach number for an airfoil of specific thickness ratio, however, depend on which relation is selected for k. Wherever such values are given, they will be those obtained by use of the expression for k given in equation (18), that is,

$$k=M_o{}^2\frac{\gamma+1}{U_o} \tag{18}$$

The principal reason for this choice is that it appears to provide a set of equations, or a mathematical model, which approximates certain essential features of transonic flow with superior accuracy.

RELATION BETWEEN TRANSONIC THEORY AND LINEAR THEORY

It is important to recognize that wing theory based on equation (7) is valid for all Mach numbers below the hypersonic range. At subsonic and supersonic speeds, equation (7) is of the same order of accuracy as the Prandtl-Glauert equation of liner theory (eq. (5)) although more difficult to solve. At $M_o=1$, equation (7) is identical with equation (25), now widely used in the study of transonic-flow problems.

On the other hand, there is no a priori method for determining whether or not a solution of the equations of linear theory will be valid in the transonic range. One can only decide by solving the problem under the assumptions of linear theory and then inspecting the magnitudes of the terms, particularly of φ_x, to see whether or not they can be regarded as small quantities. If the terms are sufficiently small, the linear-theory solution is presumed valid even though the Mach number may be near unity. Linearized-theory solutions have been obtained for a great number of practical wing problems and their behavior in the transonic range is now well known. To review briefly: For unswept wings of infinite span, linear theory indicates that the magnitude of φ_x on the surface of a given airfoil is proportional to $1/\sqrt{|1-M_o{}^2|}$; consequently, φ_x approaches infinity as M_o approaches unity and the theory is clearly inapplicable. For wings of finite span, however, the perturbation velocities may be large or small at sonic velocity, depending on the particular problem as discussed in detail in reference 26. Specifically, for three-dimensional lifting surfaces of zero

thickness, the velocities remain finite everywhere except at the leading edges, their magnitudes generally increasing with increasing aspect ratio and angle of attack. For wings of nonzero thickness, however, φ_x generally becomes large logarithmically as $1-M_o^2$ approaches zero; consequently, linear theory is inapplicable within some Mach number range surrounding unity.

Summarizing, linear theory is applicable to lifting surfaces of small or moderate aspect ratio at all transonic speeds, but fails for wings of finite thickness within a range of Mach numbers surrounding unity. The range of inapplicability diminishes to zero as the aspect ratio, thickness ratio, and angle of attack of the wing tend to zero.

In treating transonic flows for which linear theory is applicable, it is often advantageous to consider the special case of sonic flow ($M_o=1$) separately. Equation (5) for the perturbation potential then reduces to a particularly simple form

$$\varphi_{yy}+\varphi_{zz}=0$$

Solutions of this equation, in conjunction with the boundary conditions given by equations (13) and (15), are identical to those of linear theory found by solving equation (5) and subsequently setting $M_o=1$, but can be obtained with much less effort. Since, in addition, the results of this simple theory, now generally known as slender-wing theory, are also applicable to low-aspect-ratio lifting surfaces throughout the entire Mach number range, a considerable number of solutions of slender-wing theory have been presented in the last few years (e. g., réfs. 27, 28, and 29). These results are, of course, applicable to flows at $M_o=1$ to exactly the same extent as the results of linear theory.

SIMILARITY RULES

In reference 6, von Kármán derived similarity rules for the pressure distribution, lift, drag, and pitching moment of airfoils in transonic flow using equations (24) through (28). The same equations were used in reference 20 to determine the transonic similarity rules for wings of finite span. The corresponding similarity rules of linearized subsonic and supersonic wing theory were also derived and compared with the transonic similarity rules in the latter reference. It was shown that the similarity rules of linear theory contain an arbitrary parameter and can be expressed in many forms, one of which coincides with the similarity rules of transonic flow.

A derivation of the transonic similarity rules, based on the U_o equations with unspecified k, is provided in Appendix B. This derivation possesses the advantage of being based on a single statement of the problem of wing theory that is uniformly valid at subsonic, transonic, and supersonic speeds. It follows from the preceding discussion that the results so found are identical to those of references 6 and 20 if k is equated to $(\gamma+1)/U_o$. The similarity rules for C_p, C_L, C_m, and C_D are given in Appendix B as follows:

$$C_p=\frac{\tau^{2/3}}{(U_ok)^{1/3}}\,\mathcal{P}\left[\frac{\sqrt{1-M_o^2}}{(U_ok\tau)^{1/3}},\sqrt{1-M_o^2}\,A;\frac{x}{c},\frac{y}{b}\right] \quad (36)$$

$$C_L=\frac{\tau^{2/3}}{(U_ok)^{1/3}}\,\mathcal{L}\left[\frac{\sqrt{1-M_o^2}}{(U_ok\tau)^{1/3}},\sqrt{1-M_o^2}\,A\right] \quad (37)$$

$$C_m=\frac{\tau^{2/3}}{(U_ok)^{1/3}}\,\mathcal{M}\left[\frac{\sqrt{1-M_o^2}}{(U_ok\tau)^{1/3}},\sqrt{1-M_o^2}\,A\right] \quad (38)$$

$$C_D=\frac{\tau^{5/3}}{(U_ok)^{1/3}}\,\mathcal{D}\left[\frac{\sqrt{1-M_o^2}}{(U_ok\tau)^{1/3}},\sqrt{1-M_o^2}\,A\right] \quad (39)$$

where the geometry of related wings is given by equation (16):

$$(Z/c)=\tau f(x/c,\,y/b)$$

Equations (36) through (39) are functional equations. For example, equation (36) is to be interpreted as stating that the pressure coefficient C_p is equal to $\tau^{2/3}/(U_ok)^{1/3}$ times some function $\mathcal{P}$ of a number of specified parameters. The foregoing equations have been written for flows where $M_o\leq1$. If $M_o\geq1$, the radical $\sqrt{1-M_o^2}$ should be replaced with $\sqrt{M_o^2-1}$. The functions $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{D}$ are different, however, for subsonic and supersonic flow. Consequently, subsonic flows may be related to other subsonic flows by the similarity rules, but not to supersonic flows, and conversely.

ALTERNATIVE FORMS OF THE SIMILARITY RULES

It is important to recognize that the similarity parameters may be combined or regrouped in any manner whatsoever, provided the same number of independent parameters is always retained. For instance, in much of what follows, it will be found desirable to use the square of $\sqrt{1-M_o^2}/(U_ok\tau)^{1/3}$ and to replace $\sqrt{1-M_o^2}\,A$ with a new parameter $(U_ok\tau)^{1/3}A$ obtained by dividing $\sqrt{1-M_o^2}\,A$ by $\sqrt{1-M_o^2}/(U_ok\tau)^{1/3}$. In terms of these parameters, the similarity rules are

$$C_p=\frac{\tau^{2/3}}{(U_ok)^{1/3}}\,\mathcal{P}\left[\frac{M_o^2-1}{(U_ok\tau)^{2/3}},(U_ok\tau)^{1/3}A;\frac{x}{c},\frac{y}{b}\right] \quad (40)$$

$$C_L=\frac{\tau^{2/3}}{(U_ok)^{1/3}}\,\mathcal{L}\left[\frac{M_o^2-1}{(U_ok\tau)^{2/3}},(U_ok\tau)^{1/3}A\right] \quad (41)$$

$$C_m=\frac{\tau^{2/3}}{(U_ok)^{1/3}}\,\mathcal{M}\left[\frac{M_o^2-1}{(U_ok\tau)^{2/3}},(U_ok\tau)^{1/3}A\right] \quad (42)$$

$$C_D=\frac{\tau^{5/3}}{(U_ok)^{1/3}}\,\mathcal{D}\left[\frac{M_o^2-1}{(U_ok\tau)^{2/3}},(U_ok\tau)^{1/3}A\right] \quad (43)$$

wherein the geometry of related wings is again given by equation (16).

The similarity rules thus formulated are totally equivalent to those given by equations (36) through (39) but possess three outstanding advantages:

(a) The indeterminacy at $M_o=1$ resulting from two parameters simultaneously vanishing is removed.

(b) The squaring of the first parameter avoids the necessity of changing parameters as sonic speed is passed.

(c) The use of the parameter $(U_o k \tau)^{1/3} A$ rather than $\sqrt{1-M_o^2} A$ aids in distinguishing the regimes in which linear theory is applicable in the transonic range from those in which nonlinear theory must be used. Thus as $(U_o k \tau)^{1/3} A$ approaches zero, linear theory is always applicable, provided, in some cases, that M_o is not precisely equal to unity. On the other hand, as $(U_o k \tau)^{1/3} A$ becomes large, linear theory is not applicable in the transonic range and nonlinear theory must be used.

The forms of the similarity rules given in equations (36) through (43) have been the source of some confusion, due to the multiple role that τ plays in determining the thickness ratio, camber, and angle of attack. As can be seen from equation (16), all three of these geometrical quantities are linearly proportional to τ. A more explicit statement of the symmetrical profiles of nonzero thickness may be obtained by rewriting the expression for Z/c in terms of the thickness ration t/c and angle of attack α rather than τ, thus,

$$Z/c = t/c \left[g\left(\frac{x}{c}, \frac{y}{b}\right) - \frac{\alpha}{t/c} \frac{x}{c} \right] \qquad (44)$$

Comparison of equations (16) and (44) indicates that t/c plays a similar role to τ but that it is necessary to introduce a second parameter $\alpha/(t/c)$. In this way, a new set of equations expressing the similarity rules is obtained which correspond to equations (40) through (43), although expressed in terms of α and t/c rather than τ. They are

$$\tilde{C}_p = \frac{(U_o k)^{1/3}}{(t/c)^{2/3}} C_p = \mathcal{P}(\xi_o, \tilde{A}, \tilde{\alpha}; x/c, y/b) \qquad (45)$$

$$\tilde{C}_L = \frac{(U_o k)^{1/3}}{(t/c)^{2/3}} C_L = \mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (46)$$

$$\tilde{C}_m = \frac{(U_o k)^{1/3}}{(t/c)^{2/3}} C_m = \mathcal{M}(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (47)$$

$$\tilde{C}_D = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} C_D = \mathcal{D}(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (48)$$

where ξ_o, $\tilde{A}$, and $\tilde{\alpha}$ are similarity parameters defined as follows:

$$\xi_o = \frac{M_o^2 - 1}{[(U_o k)(t/c)]^{2/3}}, \quad \tilde{A} = [(U_o k)(t/c)]^{1/3} A, \quad \tilde{\alpha} = \frac{\alpha}{t/c} \qquad (49)$$

SLOPE OF PRESSURE CURVE AT $M_o=1$

Liepmann and Bryson (refs. 17 and 18) have made the following observations which enable the determination by simple and intuitive considerations of the slope of the C_p versus M_o curve at $M_o=1$. It is a well-known fact that, at slightly supersonic Mach numbers, the detached bow wave is far away from the airfoil and nearly normal. It is also well known that the Mach number downstream of weak normal shock is as much below unity as the Mach number upstream is above unity. Consequently, the Mach number distribution on the airfoil should be independent of Mach number in the neighborhood of $M_o=1$, that is,

$$\left(\frac{dM}{dM_o}\right)_{M_o=1} = 0 \qquad (50)$$

Now, if the isentropic flow relation for C_p in terms of M and M_o

$$C_p = \frac{2}{\gamma M_o^2} \left[-1 + \left(\frac{1 + \frac{\gamma-1}{2} M_o^2}{1 + \frac{\gamma-1}{2} M^2} \right)^{\frac{\gamma}{\gamma-1}} \right] \qquad (51)$$

is differentiated with respect to M_o, M_o is equated to unity, and equation (50) is introduced, the following relation, first given in reference 8 by Vincenti and Wagoner, is found:

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1} = \frac{4}{\gamma+1} - \frac{2}{\gamma+1} (C_p)_{M_o=1} \qquad (52)$$

Since the above-mentioned derivation is based to a certain extent on physical reasoning and makes use of the exact rather than approximate relation between pressure and Mach number, it is of interest from the present point of veiw to review the equivalent result contained in the model of transonic flow provided by equations (7), (12), (13), (15), and (17). In common with the other characteristics of transonic theory, the result can be expressed conveniently in the form of a similarity rule. Thus, recall the approximate relation for the local Mach number given by equation (31).

$$1 - M^2 = 1 - M_o^2 + \frac{kU_o}{2} C_p \qquad (31)$$

Now, according to the similarity rules, transonic theory does not provide information about C_p or M_o alone, but only about parameters such as $\widetilde{C_p}$ and ξ_o. Thus, the similarity rule for local Mach number is the following:

$$\xi = \frac{M^2 - 1}{[(U_o k)(t/c)]^{2/3}} = \frac{M_o^2 - 1}{[(U_o k)(t/c)]^{2/3}} - \frac{(U_o k)^{1/3}}{2(t/c)^{2/3}} C_p = \xi_o - \frac{1}{2} \widetilde{C_p} \qquad (53)$$

The solutions of the equations for transonic flow obtained for slightly supersonic flow by Vincenti and Wagoner in reference 8 and for slightly subsonic flow by Cole in reference 11 indicate that the approximate relation which corresponds to the Mach number freeze is the following:

$$\left(\frac{d\xi}{d\xi_o}\right)_{\xi_o=0} = 0 \qquad (54)$$

It may be recognized that this relation is equivalent to the exact relation given in equation (50) if k is independent of M_o, as in equation (19). This is not the case, however, when the presently preferred relation for k, namely, equation (18), is used. Given equation (54), differentiation of equation (53) shows that the slope of the pressure curve in the reduced parameters is

$$\left(\frac{d\widetilde{C_p}}{d\xi_o}\right)_{\xi_o=0} = 2 \qquad (55)$$

It is a simple matter to derive the corresponding slope for the C_p versus M_o curve at $M_o=1$, provided a specific expres-

sion is selected for k. If k is equated to $M_o^2(\gamma+1)/U_o$, the final result is

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1}=\frac{4}{\gamma+1}-\frac{2}{3}(C_p)_{M_o=1} \qquad (56)$$

It is interesting to note that if k is independent of M_o, the slope of the pressure curve is

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1}=\frac{4}{\gamma+1} \qquad (57)$$

Thus, although equation (19) leads to the exact relation for the rate of change of the local Mach number, the slope of the pressure curve is considerably less accurate than that provided by equation (18).

The range of problems to which the foregoing results apply is not known at present. Inasmuch as the entire phenomenon appears to be connected with the presence of a detached bow wave which stands normal to the flow, intuitive considerations suggest that the results are probably applicable at least to symmetrical airfoils at zero or infinitesimal angles of attack but perhaps not to airfoils at larger angles of attack or to wings of finite span.

APPLICATIONS

FUNDAMENTAL HYPOTHESES AND PRINCIPLES

The remainder of this report is principally concerned with the deduction of the qualitative, and to some extent quantitative, characteristics of thin wings in transonic flow by means of simple logical considerations based primarily on the similarity rules together with the following hypotheses:

(a) Nonlinear theory based on equation (7) is applicable to all problems.

(b) Linear theory based on equation (5) is valid for all wings at Mach numbers either appreciably below or above unity.

(c) Linear theory is valid at all Mach numbers, except possibly very near unity, for wings of small aspect ratio.

(d) The differential pressures between the upper and lower surfaces of a wing having symmetrical airfoil sections are proportional to the angle of attack for at least a small range of angles about zero.

(e) The slope of $\widetilde{C_p}$ versus ξ_o at $M_o=1$, defined by equation (55), is applicable at least to symmetrical airfoil sections at zero or infinitesimal angles of attack.

The consequences of the foregoing statements will be consistently pursued in the following sections in the discussion of the aerodynamic characteristics of airfoils and complete wings. Throughout, the analysis will be restricted to wings having symmetrical profiles. Whenever specific results are to be used to illustrate the statements, they will nearly always be for symmetrical-wedge or double-wedge profiles and for wings of triangular plan form. This choice is dictated by the availability of theoretical results.

The basic principle in the following analysis is to express the similarity rules in such forms that the lift, pitching-moment, and drag coefficients can be studied for limiting values of the parameters with no chance for ambiguity due to indeterminate forms. In this respect, the statement of the similarity

rules provided by equations (45) through (49) will be found particularly useful.

PRESSURE DRAG OF SYMMETRICAL NONLIFTING WINGS

The similarity rule for the pressure drag coefficient of symmetrical nonlifting wings having profiles given by

$$(Z/c=(t/c)\,g(x/c,\,y/b) \qquad (58)$$

is obtained from equations (44) and (48) by setting α, the angle of attack, to zero

$$\widetilde{C}_{D_o}=\frac{(U_ok)^{1/3}}{(t/c)^{5/3}}\,C_{D_o}=\mathcal{D}_o\,(\xi_o,\tilde{A}) \qquad (59)$$

where

$$\xi_o=\frac{M_o^2-1}{[(U_ok)\,(t/c)]^{2/3}},\quad \tilde{A}=[U_ok\,(t/c)]^{1/3}\,A \qquad (49)$$

Therefore, drag results for symmetrical nonlifting wings should be presented by plotting the variation of the generalized drag coefficient $\widetilde{C_{D_o}}$ with ξ_o and $\tilde{A}$.

At Mach numbers sufficiently removed from unity for linear theory to apply, C_{D_o} must be independent of k since k does not appear in either the differential equation or boundary conditions of linear theory. This implies that the parameters ξ_o and $\tilde{A}$ must be arranged in such a manner that k is canceled completely from the above relation for C_{D_o}. The only alternative inside the function $\mathcal{D}_o$ is to form the product $\sqrt{\xi_o}\,\tilde{A}$. Numerous possibilities exist on the outside of $\mathcal{D}_o$ depending on whether k is canceled by using ξ_o, $\tilde{A}$, or some combination thereof. Although any of these are acceptable, the first will be preferred. In this way the following results are obtained:

$$(C_{D_o})_l=\frac{(t/c)^{5/3}}{(U_ok)^{1/3}}\left\{\frac{|M_o^2-1|}{[(U_ok)\,(t/c)]^{2/3}}\right\}^{-1/2}\mathcal{D}_{o_l}\left[\left\{\frac{|M_o^2-1|}{[(U_ok)\,(t/c)]^{2/3}}\right\}^{1/2}\times\right.$$
$$\left.[(U_ok)\,(t/c)]^{1/3}\,A\right]=\frac{(t/c)^2}{\sqrt{|M_o^2-1|}}\,\mathcal{D}_{o_l}\,(\sqrt{|M_o^2-1|}A) \qquad (60)$$

where it should be recalled that $\mathcal{D}_{o_l}$ is a different function for subsonic and supersonic flow. Equation (60) is equivalent to the extended Prandtl-Glauert rule. For subsonic flow, D'Almbert's paradox requires that the drag be zero; therefore, for all wings,

$$(C_{D_o})_l=0,\qquad (\widetilde{C_{D_o}})_l=0 \qquad (61)$$
$$\scriptstyle M_o<1 \qquad\qquad \xi_o<0$$

For supersonic flow, wave drag exists which depends on $\sqrt{M_o^2-1}\,A$ as well as on the plan form and airfoil section. The general functional relation for the drag coefficient of a family of affinely related wings at zero angle of attack, as given by linear theory, is

$$(C_{D_o})_l=\frac{(t/c)^2}{\sqrt{M_o^2-1}}\mathcal{D}_{o_l}(\sqrt{M_o^2-1}A),\ (\widetilde{C_{D_o}})_l=\xi_o^{-1/2}\mathcal{D}_{o_l}(\xi_o^{1/2}\tilde{A})$$
$$\scriptstyle M_o>1 \qquad\qquad\qquad\qquad \xi_o>0 \qquad (62)$$

Wings of infinite aspect ratio.—For wings of infinite span (or airfoils), equation (62), representing the functional

relation of linear theory for the drag coefficient, reduces to the following:

$$(c_{d_o})_{\substack{l \\ M_o>1}} = \frac{(t/c)^2}{\sqrt{M_o{}^2-1}} \times \text{const.}, \qquad (\widetilde{c_{d_o}})_{\substack{l \\ \xi_o>0}} = \frac{\text{const.}}{\sqrt{\xi_o}} \qquad (63)$$

where the value of the constant depends on the shape of the airfoil. Numerous experimental data show variations consistent with equation (63) at Mach numbers greater than about that of shock attachment. At Mach numbers closer to unity, however, the theoretical values provided by this equation are unreliable. It is evident from inspection of the results of linear theory itself that such a failure occurs, since the perturbation velocities assumed to be small in the derivation of the equations are found to become infinitely large as the Mach number approaches unity. It is apparent, therefore, that it is necessary to resort to nonlinear theory for the calculation of the drag of airfoils in the transonic speed range.

A similarity rule for the generalized section drag coefficient of symmetrical nonlifting airfoils which is valid throughout the Mach number range may be obtained from equation (59) by setting $\tilde{A}$ equal to infinity.

$$\widetilde{c_{d_o}} = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} c_{d_o} = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}} (C_{D_o})_{\tilde{A}=\infty} = \mathcal{D}_o(\xi_o, \infty) = d_o(\xi_o) \quad (64)$$

At a Mach number of unity, the similarity parameter ξ_o vanishes and the function $d_o(\xi_o)$ is a constant

$$(\widetilde{c_{d_o}})_{\xi_o=0} = d_o(0) = \text{const.}, \qquad (c_{d_o})_{M_o=1} = \frac{(t/c)^{5/3}}{(U_o k)^{1/3}} \times \text{const.} \quad (65)$$

indicating that the section drag coefficients of affinely related airfoils are proportional to the five-thirds power of their thickness ratios. If hypothesis (e) is accepted, the variation of $\widetilde{c_{d_o}}$ with ξ_o at $M_o=1$ is found to be zero for completed airfoils

$$\left(\frac{d\widetilde{c_{d_o}}}{d\xi_o}\right)_{\xi_o=0} = \oint \left(\frac{d\widetilde{C_p}}{d\xi_o}\right)_{\xi_o=0} \left[\frac{d(Z/t)}{d(x/c)}\right] d\frac{x}{c} = 2\oint \frac{d(Z/t)}{d(x/c)} d\frac{x}{c} = 0 \quad (66)$$

If k is taken to be $M_o{}^2(\gamma+1)/U_o$, $(dC_p/dM_o)_{M_o=1}$ is given by equation (56) and the slope of the drag curve is

$$\left(\frac{dc_{d_o}}{dM_o}\right)_{M_o=1} = \oint \left(\frac{dC_p}{dM_o}\right)_{M_o=1} \frac{dZ}{dx} d\left(\frac{x}{c}\right)$$

$$= \oint \left[\frac{4}{\gamma+1} - \frac{2}{3}(C_p)_{M_o=1}\right] \frac{dZ}{dx} d\left(\frac{x}{c}\right) = -\frac{2}{3}(c_{d_o})_{M_o=1}$$

$$(67)$$

The corresponding exact relation can be determined similarly by use of equation (52) for the slope of the pressure curve. It is

$$\left(\frac{dc_{d_o}}{dM_o}\right)_{M_o=1} = \oint \left[\frac{4}{\gamma+1} - \frac{2}{\gamma+1}(C_p)_{M_o=1}\right] \frac{dZ}{dx} d\left(\frac{x}{c}\right)$$

$$= -\frac{2}{\gamma+1}(c_{d_o})_{M_o=1} \quad (68)$$

Since calculations have been made of the drag in transonic flow of simple symmetrical sections at zero angle of attack, it is not necessary to speculate further regarding the variation of $\widetilde{c_{d_o}}$ with ξ_o. At present, complete theoretical information exists for the drag of symmetrical double-wedge airfoils throughout the transonic range. Solutions for this section have been obtained by transforming the nonlinear differential equation of the small disturbance transonic theory into hodograph variables and taking advantage of simplification of the normally difficult problems relating to the boundary conditions by restricting attention to polygonal profiles. The results for flows having subsonic, sonic, and supersonic free-stream velocities have been given, respectively, by Trilling (ref. 12), Guderley and Yoshihara (ref. 7), and Vincenti and Wagoner (ref. 8). The linear-theory solution for pure supersonic flows has been given by Ackeret (ref. 30). All of these results are combined on a single graph in figure 5. It may be seen that the preceding
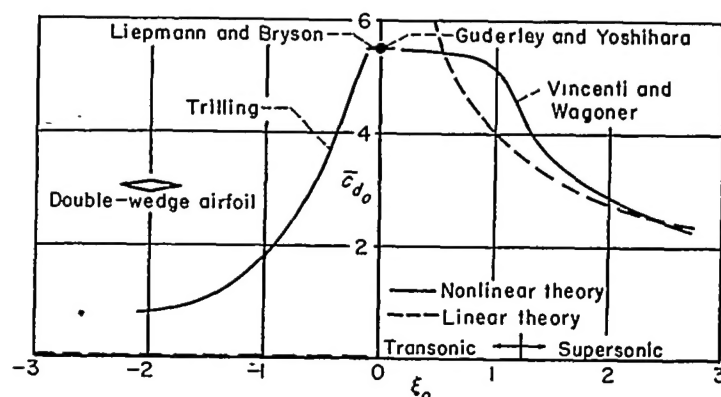

FIGURE 5.—Theoretical drag of double-wedge airfoil.

remarks concerning the relation of the results of linear theory and nonlinear theory, and the slope of the drag curve at a Mach number of unity are illustrated by this comparison.

Osawatitsch has given approximate solutions for $M_o<1$ for the pressure distribution on symmetrical biconvex airfoils (refs. 13 and 14) and for the pressure distribution and drag on NACA four-digit symmetrical airfoils (ref. 14). This work has recently been extended to several NACA 6-series airfoils by Gullstrand (refs. 15 and 16). Their drag results are generally similar to those indicated for the double-wedge section in figure 5.

In problems such as we are considering herein, the final test is provided by comparison with experimental results. Unfortunately, experimental results for the transonic speed range are scarce, as well as difficult to obtain, and no data are available for direct comparison with the theoretical results summarized in figure 5. At present, however, complete information, both theoretical and experimental, does exist for a single-wedge section followed by a straight section extending far downstream. In accord with some of the original papers on this subject, the single-wedge section is considered as the front half of a symmetrical double-wedge airfoil having a chord c. Solutions for this section obtained using transonic flow theory have been given for flows having subsonic, sonic, and supersonic free-stream velocities, re-

spectively, by Cole (ref. 11), Guderley and Yoshihara (ref. 7), and Vincenti and Wagoner (ref. 8). These results are shown in figure 6 together with the corresponding experimental
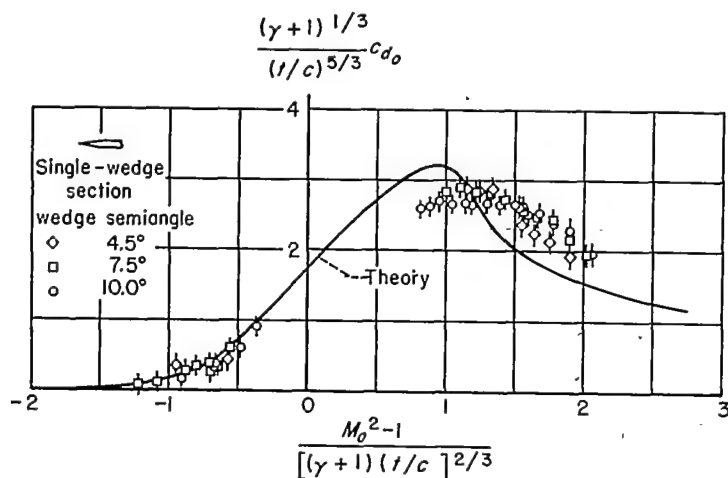

FIGURE 6.—Drag correlation of single-wedge section, $k=(\gamma+1)/U_o$.

results of Liepmann and Bryson (refs. 17 and 18). Both the theoretical curve and the experimental points are plotted in the same manner as presented in the original papers. As remarked in a preceding section, the theoretical results are based on a set of equations equivalent to the present equations with k equated to $(\gamma+1)/U_o$. The small vertical lines on the experimental data points represent the uncertainty of the values. The slope of the drag curve at $M_o=1$ is no longer zero as indicated for the complete airfoil by equation (66) but takes on a positive value given originally by Liepmann and Bryson (ref. 17) and readily derivable from equation (66).

$$\left(\frac{\widetilde{dc_{d_o}}}{d\xi_o}\right)_{\xi_o=o}=2 \qquad (69)$$

This figure indicates that the theoretical and experimental results are only in general qualitative agreement. Since it is shown in a preceding section that theoretical considerations suggest that a superior theory results if k is equated to $M_o^2(\gamma+1)/U_o$ rather than $(\gamma+1)/U_o$, it is of interest to recalculate and replot the present results accordingly so as to ascertain to what degree these thoughts are borne out by actual experiment. The results are shown in figure 7. It can be seen that the theoretical and experimental results are in nearly perfect agreement. Comparison of figures 6 and 7 provides striking evidence supporting the contention that k should be equated to $M_o^2(\gamma+1)/U_o$ rather than $(\gamma+1)/U_o$, as has been done so often in the past.

Wings of finite aspect ratio.—The similarity rules for wings of finite aspect ratio are given by equations (59) and (60). Although no essential simplification of the rules occurs for wings of small aspect ratio, the range of applicability of linear theory increases as the aspect ratio decreases. This point can be illustrated by considering the results provided by linear theory in a specific case. A good example to select for this purpose is that of the drag in supersonic flow of a triangular wing with symmetrical double-wedge airfoils. (See ref. 31.) This particular choice was made for the follow-
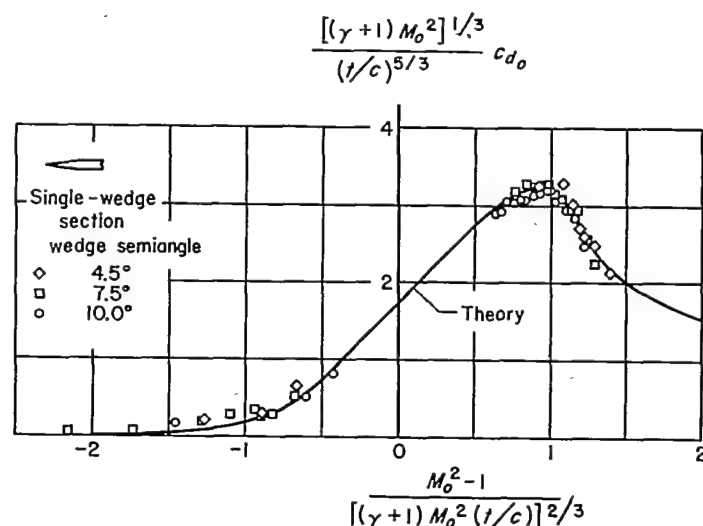
FIGURE 7.—Drag correlation of single-wedge section, $k=M_o^2(\gamma+1)/U_o$.

ing reasons: (a) Solutions are known for all supersonic Mach numbers; (b) the double-wedge airfoil discussed in the preceding sections corresponds to the limiting case of the wing of very great aspect ratio. The drag results provided for this wing by linear theory are presented in figure 8.
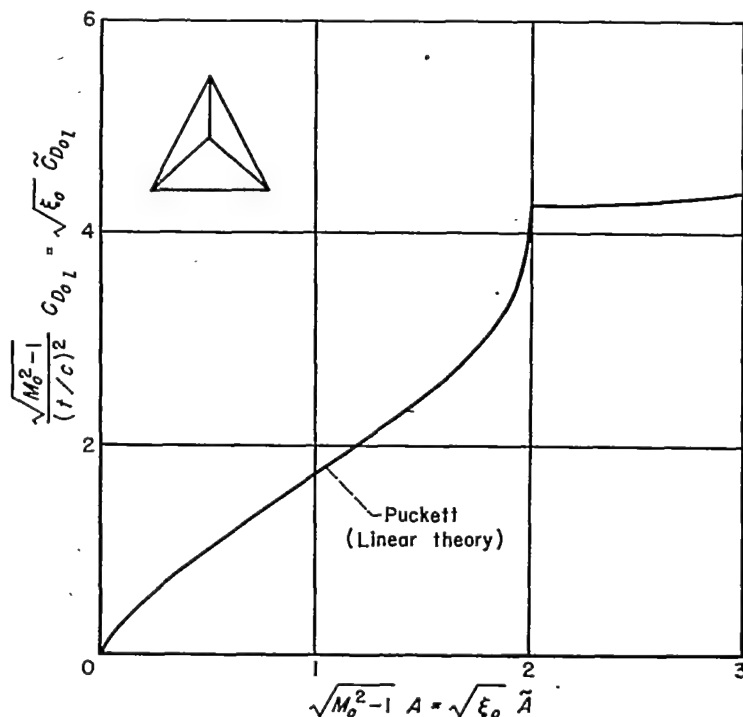

FIGURE 8.—Supersonic drag of triangular wing, linear theory.

The results of figure 8 are presented in a different manner in figure 9 wherein $\widetilde{C_{D_o}}$ is plotted as a function of ξ_o for various values of $\tilde{A}$ as suggested by equation (59). For purposes of comparison, the curves for the drag of airfoils ($\tilde{A}=\infty$) computed by both linear and nonlinear theory are also included in the graph. As noted in the preceding section on airfoils, comparison of the results of linear theory with those of nonlinear theory for wings of $\tilde{A}=\infty$ shows that good agreement exists for larger ξ_o, but that at smaller
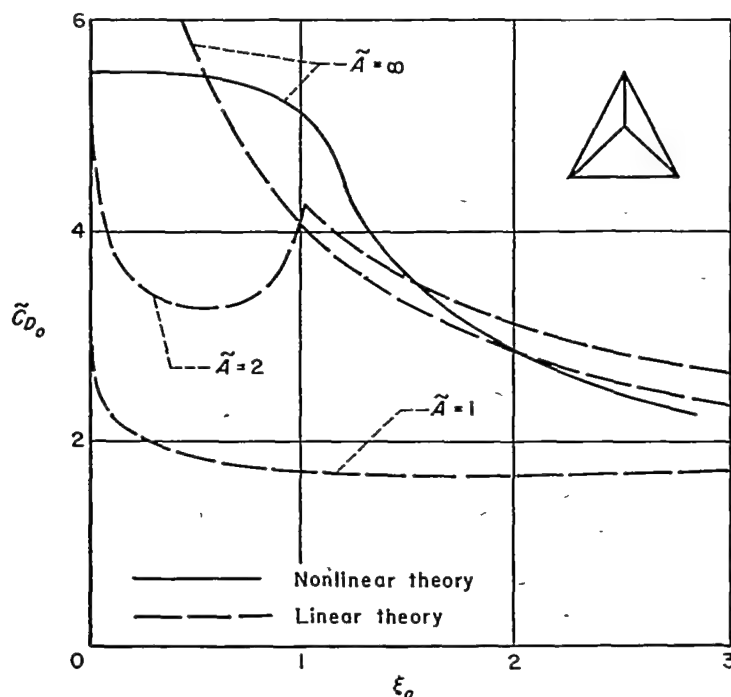
FIGURE 9.—Variation with ξ_o of reduced drag coefficient of triangular wing.

FIGURE 10.—Variation with $\tilde{A}$ of reduced drag coefficient of triangular wings.

ξ_o, the values predicted by linear theory become too large. For wings of finite $\tilde{A}$, only the results of linear theory are available. They also exhibit the trend of indicating infinite drag as ξ_o approaches zero; however, the range of ξ_o in which the value of $\widetilde{C_{D_o}}$ is excessive is much less than is the case for $\tilde{A}=\infty$. In general, $\widetilde{C_{D_o}}$ of wings of small aspect ratio diminishes with decreasing aspect ratio.

The drag results of figure 8 are presented in still another form in figure 10 in which is plotted the variation of $\widetilde{C_{D_o}}$ with $\tilde{A}$ for various values of ξ_o. The principal merit of this method of plotting is that it aids in distinguishing the region where nonlinear theory must be used from that where linear theory may be useful. Thus, the two-dimensional nonlinear theory results appear on the right of the graph corresponding to large $\tilde{A}$; whereas the three-dimensional linear theory results appear on the left for small $\tilde{A}$. The filling in of the remainder of the graph requires either the solution of the equations of three-dimensional nonlinear wing theory or the use of experimental data. It should again be noted that the present drag considerations apply only to the pressure drag. Before plotting experimental results in the manner indicated, it is necessary to first subtract the friction drag.

LIFT

The similarity rule for the lift coefficient C_L of a family of wings having ordinates Z given by equation (44)

$$Z/c = t/c \left[g(x/c, y/b) - \frac{\alpha}{t/c} \frac{x}{c} \right] \qquad (44)$$

is given in equation (46) as

$$\widetilde{C_L} = \frac{(U_o k)^{1/3}}{(t/c)^{2/3}} \, C_L = \mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (46)$$
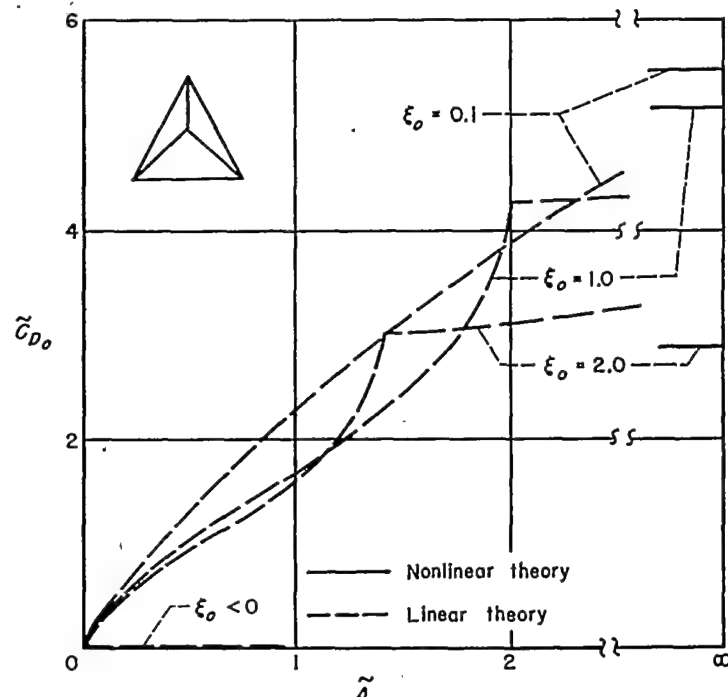
where

$$\xi_o = \frac{M_o^2 - 1}{[U_o k(t/c)]^{2/3}}, \quad \tilde{A} = [U_o k(t/c)]^{1/3} A, \quad \tilde{\alpha} = \frac{\alpha}{t/c} \qquad (49)$$

If hypothesis (d) is accepted, C_L varies linearly with α for at least small angles of attack. It is advantageous, therefore, to consider the lift ratio C_L/α rather than C_L alone, thereby minimizing the influence of $\tilde{\alpha}$.

$$\frac{C_L}{\alpha} = \frac{1}{\alpha} \frac{(t/c)^{2/3}}{(U_o k)^{1/3}} \mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha}) = \frac{1}{[U_o k(t/c)]^{1/3}} \mathcal{L}'(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (70)$$

where $\mathcal{L}'(\xi_o, \tilde{A}, \tilde{\alpha})$ is a new function of the indicated variables obtained from $\mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha})$ by division by $\tilde{\alpha}$. Therefore, lift results may be presented by plotting the variation with ξ_o, $\tilde{A}$, and $\tilde{\alpha}$ of a generalized lift ratio $\widetilde{C_L/\alpha}$ defined as follows (the prime on $\mathcal{L}$ has been omitted for simplicity):

$$\widetilde{C_L/\alpha} = [U_o k(t/c)]^{1/3} (C_L/\alpha) = \mathcal{L}(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (71)$$

Equation (71) shows that $\widetilde{C_L/\alpha}$ depends upon three parameters, one more than the number which can readily be treated on a simple plot. Simplification can be gained, of course, by holding one of the parameters constant for an entire graph. Results so presented are particularly interesting for $\xi_o = 0$, $(M_o = 1)$; $\tilde{A} = \infty$, $(A = \infty)$; and $\tilde{\alpha} = 0$, $(\alpha = 0)$. The latter scheme is especially good since experiments indicate that lift curves of wings are often relatively straight lines at all Mach numbers. The values of $\widetilde{C_L/\alpha}$ at $\tilde{\alpha} = 0$ might, therefore, be expected to be good indications of the actual values for other $\tilde{\alpha}$. The appropriate similarity rule may then be written

$$(\widetilde{C_L/\alpha})_{\tilde{\alpha}=0} = \mathcal{L}(\xi_o, \tilde{A}, 0) = \mathcal{L}_o(\xi_o, \tilde{A}) \qquad (72)$$

For cases where linear theory applies (hypotheses (b) and (c)), two statements can be made immediately which provide further information about C_L/α: (a) $(C_L/\alpha)_l$ must be independent of k, and (b) $(C_L/\alpha)_l$ must be independent of $\tilde{\alpha}$ by virtue of the superposition principle of linear theory. Therefore, following the procedure used in equation (60) gives

$$\left(\frac{C_L}{\alpha}\right)_l = \frac{1}{\sqrt{|M_o{}^2-1|}} \mathcal{L}_l\left(\sqrt{|M_o{}^2-1|}A\right), \quad \left(\widetilde{\frac{C_L}{\alpha}}\right)_l = \frac{1}{\sqrt{|\xi_o|}} \mathcal{L}_l\left(\sqrt{|\xi_o|}\tilde{A}\right) \tag{73}$$

where again $\mathcal{L}_l$ is a different function for subsonic and supersonic flow.

Wings of infinite aspect ratio.—For wings of infinite aspect ratio, the functional relation of linear theory for the lift ratio given by equation (73) reduces to

$$\left(\frac{c_l}{\alpha}\right)_l = \frac{\text{const.}}{\sqrt{|M_o{}^2-1|}}, \qquad \left(\widetilde{\frac{c_l}{\alpha}}\right)_l = \frac{\text{const.}}{\sqrt{|\xi_o|}} \tag{74}$$

Solutions of the equations of linear theory show that the value of the constant is 2π for subsonic flow and four for supersonic flow. Examination of these results indicates that they are valid at Mach numbers appreciably less than or greater than unity, but are invalid for Mach numbers near 1.

A similarity rule for the section lift coefficients of a family of affinely related symmetrical airfoils which is valid throughout the Mach number range may be obtained from equation (71) by setting $\tilde{A}=\infty$.

$$\widetilde{(c_l/\alpha)} = \mathcal{L}(\xi_o, \infty, \tilde{\alpha}) = (\xi_o, l\tilde{\alpha}) \tag{75}$$

At a Mach number of unity, ξ_o is identically zero and the expression for the lift ratio becomes

$$\widetilde{(c_l/\alpha)}_{\xi_o=0} = l(0,\tilde{\alpha}), \qquad (c_l/\alpha)_{M_o=1} = \frac{1}{[U_o k(t/c)]^{1/3}} l\left(0, \frac{\alpha}{t/c}\right) \tag{76}$$

Equation (76), when considered together with hypothesis (d), indicates that at sonic speed, the lift-curve slope at zero angle of attack of airfoils of a single family varies inversely as the cube root of the thickness ratio, thus,

$$(c_l/\alpha)_{\substack{M_o=1\\\alpha=0}} = \frac{1}{[U_o k (t/c)]^{1/3}} l(0, \tilde{\alpha}\to0) = \frac{\text{const.}}{[U_o k (t/c)]^{1/3}} \tag{77}$$

Note that as the thickness ratio goes to zero, the value of the lift-curve slope at zero angle of attack becomes infinite, just as is indicated by equation (74) to be the case according to linear theory. If, on the other hand, t/c diminishes to zero while α is fixed so that $\tilde{\alpha}$ becomes very large, it is plausible that the thickness ratio does not have any effect on c_l. We thus have the following:

$$c_{l\substack{M_o=1\\t/c\to0}} = \frac{\alpha}{[U_o k (t/c)]^{1/3}} l\left(0, \frac{\alpha}{t/c}\to\infty\right) = \frac{\dot{\alpha}^{2/3}}{(U_o k)^{1/3}} \times \text{const.} \tag{78}$$

In this case, c_l is proportional to the two-thirds power of α.

If hypothesis (e) is acceptable, the variation of c_l/α with ξ_o at $M_o=1$ is zero, since

$$\left[\frac{d}{d\xi_o}\left(\widetilde{\frac{c_l}{\alpha}}\right)_{\alpha\to0}\right]_{\xi_o=0} = \oint\left[\frac{d}{d\xi_o}\left(\widetilde{\frac{C_p}{\tilde{\alpha}}}\right)_{\tilde{\alpha}\to0}\right]_{\xi_o=0} d\left(\frac{x}{c}\right)$$
$$= \oint\left(\frac{2}{\tilde{\alpha}}\right)_{\tilde{\alpha}\to0} d\left(\frac{x}{c}\right)=0 \tag{79}$$

If k is taken to be $M_o{}^2(\gamma+1)/U_o$, $(dC_p/dM_o)_{M_o=1}$ is given by equation (56) and the variation of c_l/α with M_o at $M_o=1$ is

$$\left[\frac{d}{dM_o}\left(\frac{c_l}{\alpha}\right)_{\alpha\to0}\right]_{M_o=1} = \oint\left[\frac{d(C_p/\alpha)}{dM_o}\right]_{\substack{\alpha\to0\\M_o=1}} d\left(\frac{x}{c}\right) = \oint\left[\frac{4}{\alpha(\gamma+1)} - \right.$$
$$\left.\frac{2}{3}\left(\frac{C_p}{\alpha}\right)_{\substack{\alpha\to0\\M_o=1}}\right] d\left(\frac{x}{c}\right) = -\frac{2}{3}\left(\frac{c_l}{\alpha}\right)_{\substack{\alpha\to0\\M_o=1}} \tag{80}$$

The corresponding exact relation can be determined similarly when equation (52) is used for the slope of the pressure curve. It is

$$\left[\frac{d}{dM_o}\left(\frac{c_l}{\alpha}\right)_{\alpha\to0}\right]_{M_o=1} = -\frac{2}{\gamma+1}\left(\frac{c_l}{\alpha}\right)_{\substack{\alpha\to0\\M_o=1}} \tag{81}$$

At present, calculations have been made of the lift in sonic flow (Guderley and Yoshihara, ref. 32) and in slightly supersonic flow (Vincenti and Wagoner, ref. 10) of symmetrical double-wedge airfoils inclined an infinitesimal angle of attack. The corresponding solutions for airfoils in slightly subsonic flow have not yet been found. The results of the above-mentioned investigations are shown in figure 11 together with
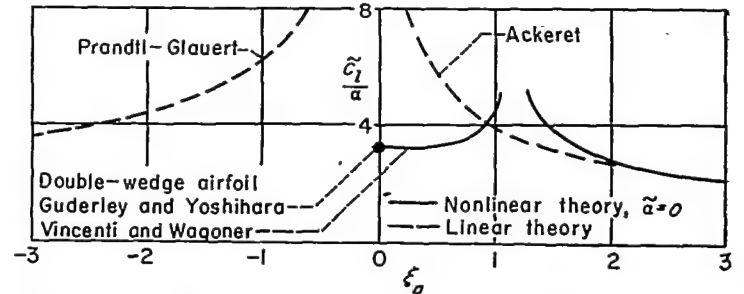

FIGURE 11.—Theoretical lift-curve slope of double-wedge airfoils.

the corresponding values given by linear theory. As in the drag case discussed in the preceding section, it may be seen that the remarks concerning the relation between the results of linear theory and nonlinear theory and the slope of the curve of c_l/α versus ξ_o at $\xi_o=0$ are verified for this particular airfoil.

Wings of vanishing aspect ratio.—Two well-known results of linear theory are that the lift-curve slopes of wings of finite aspect ratio remain finite throughout the entire Mach number range and that the lift-curve slopes of wings of vanishing aspect ratio are independent of Mach number, Therefore, equation (73) implies that $(\widetilde{C_L/\alpha})_l$ must be proportional to $\tilde{A}$ either for wings of vanishing $\tilde{A}$ in any flow, or for any wing in a flow of vanishing ξ_o:

$$(\widetilde{C_L/\alpha})_{l\atop A\to0} = (\widetilde{C_L/\alpha})_{l\atop \xi_o=0} = \tilde{A}\times\text{const.}, \quad (C_L/\alpha)_{l\atop A\to0} = (C_L/\alpha)_{l\atop M_o=1} = A\times\text{const.} \tag{82}$$

The value of the constant must be determined for each plan

form by actually solving the equations of linear theory. For wings having trailing edges which possess no cutouts extending forward of the most forward station of maximum span, that is, triangular, rectangular, elliptical, etc., as well as certain sweptback wings, the value of the constant is $\pi/2$. (Refs. 26 through 29 should be consulted for further discussion of this point as well as for the values of the constant for wings having cutouts in the trailing edge which violate the above-stated condition.)

It is seen from equation (82) that the lift-curve slopes of wings in sonic flow decrease continuously in magnitude as the aspect ratio diminishes toward zero. Since, in addition, the lift-curve slope given by linear theory has its maximum value at $M_o=1$, it is conjectured that the lift results of linear theory are a good approximation to those of nonlinear theory not only for all wings at Mach numbers far from unity but also for all Mach numbers for wings of sufficiently small aspect ratio.

Wings of finite aspect ratio.—At the present time no solutions of the nonlinear theory are available for wings of finite aspect ratio. However, from the remarks of the preceding paragraphs, it is apparent that a curve representing the variation of $\widetilde{C_L/\alpha}$ with $\tilde{A}$ for constant ξ_o and $\tilde{\alpha}$ would have the following asymptotic properties: $\widetilde{C_L/\alpha}$ would increase linearly with $\tilde{A}$ for small $\tilde{A}$ and be independent of $\tilde{A}$ for large $\tilde{A}$. In order to give a better idea of the numerical values to be expected, a set of typical results of this type is shown in figure 12 for wings having triangular plan forms and sym-
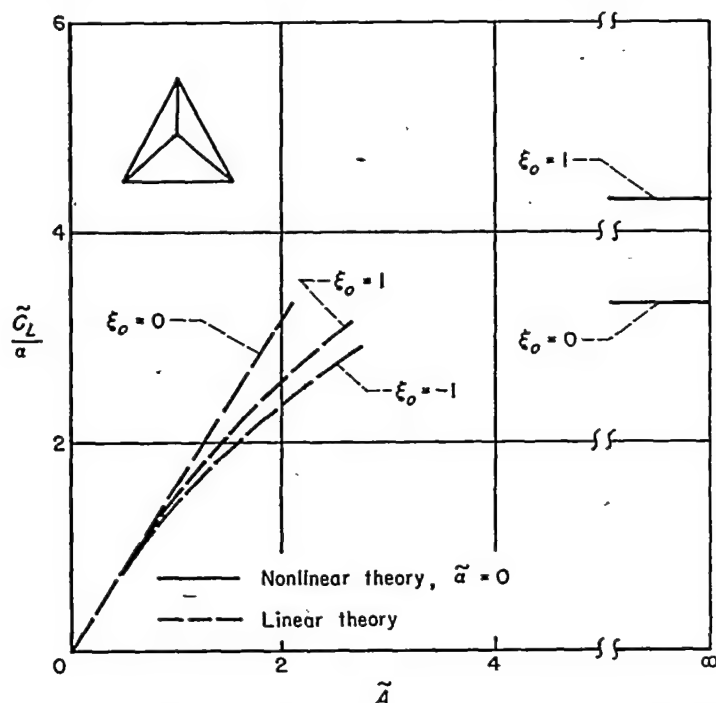

FIGURE 12.—Variation with $\tilde{A}$ of reduced lift-curve slope of triangular wings.

metrical double-wedge airfoil sections. The supersonic results are those of Stewart, Brown (refs. 33 and 34), and others. The subsonic results are those calculated by De Young and Harper (ref. 35) using Weissinger's modified lifting-line theory.

An interesting result of the foregoing remarks concerns the influence of thickness ratio on the lift-curve slope at zero angle of attack of wings in sonic flow. For wings of large aspect ratio, the lift-curve slope is inversely proportional to the cube root of the thickness ratio. For wings of small aspect ratio, the lift-curve slope is independent of the thickness ratio.

PITCHING MOMENT

The remarks of the pitching-moment characteristics of wings follow in a manner exactly analogous to those just stated for the lift characteristics. The corresponding statements for the pitching-moment coefficient C_m may be obtained by simply replacing C_L with C_m and $\mathcal{L}$ with $\mathcal{M}$. Thus, the similarity rules for C_m corresponding to equations (71) and (73) are the following, respectively:

$$(\widetilde{C_m/\alpha})=[U_o k(t/c)]^{1/3}(C_m/\alpha)=\mathcal{M}(\xi_o,\ \tilde{A},\ \tilde{\alpha}) \quad (83)$$

$$\left(\frac{C_m}{\alpha}\right)_l=\frac{1}{\sqrt{|M_o{}^2-1|}}\ \mathcal{M}_l(\sqrt{|M_o{}^2-1|}\ A),$$

$$\left(\frac{\widetilde{C_m}}{\alpha}\right)_l=\frac{1}{\sqrt{|\xi_o|}}\ \mathcal{M}_l(\sqrt{|\xi_o|}\ \tilde{A}) \quad (84)$$

where once more $\mathcal{M}$ and $\mathcal{M}_l$ are different functions for subsonic and supersonic flow. The only difference between the discussion of C_m and C_L is that the values of the constants of equations (74), (77), (78), and (82) are, of course, different. Graphs of theoretical pitching-moment characteristics for airfoils and for triangular wings corresponding to the lift results of figures 11 and 12 are shown in figures 13 and 14.
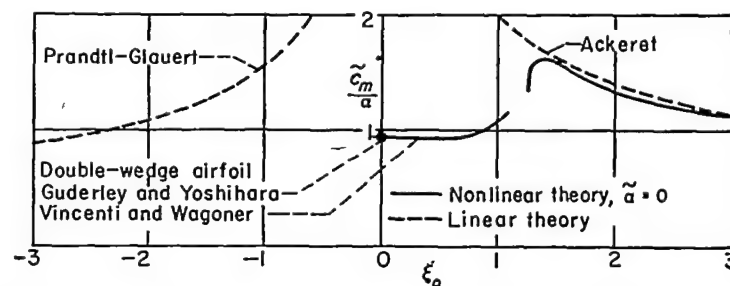

FIGURE 13.—Theoretical pitching-moment curve slope of double-wedge airfoils.

The moment axis is taken to be through the most forward point of the wing.

Sometimes it is desired to present pitching-moment characteristics of wings in terms of center-of-pressure position rather than pitching-moment coefficient. Since the center-of-pressure position can be expressed in terms of C_m and C_L by

$$\frac{x_{c.p.}}{c}=-\frac{C_m}{C_L} \quad (85)$$

the resulting expression for the center-of-pressure position found through application of equations (71) and (83) is

$$\frac{x_{c.p.}}{c}=-\frac{\mathcal{M}(\xi_o,\ \tilde{A},\ \tilde{\alpha})}{\mathcal{L}(\xi_o,\ \tilde{A},\ \tilde{\alpha})}=C.P.(\xi_o,\ \tilde{A},\ \tilde{\alpha}) \quad (86)$$
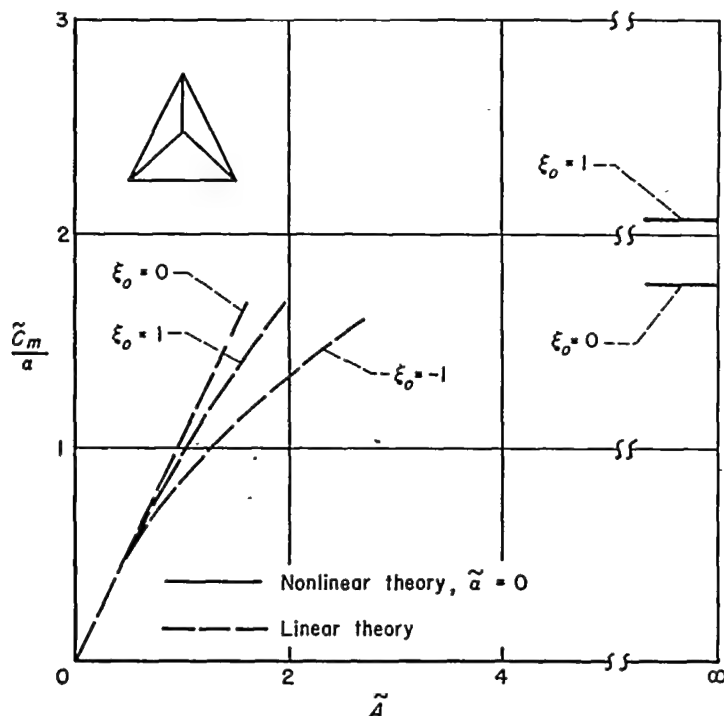
FIGURE 14.—Variation with $\tilde{A}$ of reduced pitching-moment curve slope of triangular wings.

The corresponding relation for linear theory is

$$\left(\frac{x_{o.p.}}{c}\right)_l = C.P._l\left(\sqrt{|M_o{}^2-1|}\,A\right) = C.P._l\left(\sqrt{|\xi_o|}\,\tilde{A}\right) \qquad (87)$$

PRESSURE DRAG DUE TO LIFT

The similarity rule for the pressure drag of inclined wings having symmetrical airfoils is indicated by equation (48) to be the following:

$$\widetilde{C_D} = \frac{(U_o k)^{1/3}}{(t/c)^{5/3}}\,C_D = \mathcal{D}(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (88)$$

The portion of the drag due to lift is therefore

$$\Delta C_D = C_D - C_{D_o} = \frac{(t/c)^{5/3}}{(U_o k)^{1/3}}[\mathcal{D}(\xi_o, \tilde{A}, \tilde{\alpha}) - \mathcal{D}(\xi_o, \tilde{A}, 0)]$$

$$= \frac{(t/c)^{5/3}}{(U_o k)^{1/3}}\mathcal{D}_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (89)$$

Since the differential pressures between upper and lower surfaces of wings having symmetrical airfoil sections are proportional to α for at least a small range of α surrounding zero (hypothesis (d)), it follows that ΔC_D is proportional to the square of α. It is advantageous, therefore, to consider the drag-rise ratio $\Delta C_D/\alpha^2$ rather than ΔC_D alone, thus,

$$\frac{\Delta C_D}{\alpha^2} = \frac{1}{\alpha^2}\frac{(t/c)^{5/3}}{(U_o k)^{1/3}}\mathcal{D}_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) = \frac{1}{[U_o k(t/c)]^{1/3}}\mathcal{D}_\Delta{}'(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (90)$$

where $\mathcal{D}_\Delta{}'$ is a new function of the indicated variables obtained from $\mathcal{D}_\Delta$ by dividing by $\tilde{\alpha}^2$. Consequently, drag-due-to-lift data should be presented by plotting the variation with ξ_o, $\tilde{A}$, and $\tilde{\alpha}$ of a generalized drag-rise ratio $\widetilde{\Delta C_D/\alpha^2}$ defined as follows (the prime on $\mathcal{D}_\Delta$ is omitted for simplicity):

$$\widetilde{\Delta C_D/\alpha^2} = [U_o k(t/c)]^{1/3}(\Delta C_D/\alpha^2) = \mathcal{D}_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) \qquad (91)$$

The actual presentation of the results of this three-parameter system may be accomplished as described in the section on the lift of wings. Of particular interest is the simplification resulting from presenting only the values found at $\alpha=0$. The simplified similarity rule is then

$$\widetilde{\left(\Delta C_D/\alpha^2\right)}_{\tilde{\alpha}=0} = \mathcal{D}_\Delta(\xi_o, \tilde{A}, 0) \qquad (92)$$

For cases where linear theory applies, the following results hold:

$$\left(\frac{\Delta C_D}{\alpha^2}\right)_l = \frac{1}{\sqrt{|M_o{}^2-1|}}\mathcal{D}_{\Delta_l}(\sqrt{|M_o{}^2-1|}A),$$

$$\widetilde{\left(\frac{\Delta C_D}{\alpha^2}\right)}_l = \frac{1}{\sqrt{|\xi_o|}}\mathcal{D}_{\Delta_l}(\sqrt{|\xi_o|}\,\tilde{A}) \qquad (93)$$

Wings of infinite aspect ratio.—For wings of infinite aspect ratio, the functional relation of linear theory for the drag due to lift, equation (93), reduces to

$$\left(\frac{\Delta c_d}{\alpha^2}\right)_l = \frac{\text{const.}}{\sqrt{|M_o{}^2-1|}}, \qquad \widetilde{\left(\frac{\Delta c_d}{\alpha^2}\right)}_l = \frac{\text{const.}}{\sqrt{|\xi_o|}} \qquad (94)$$

Solutions of the equations of linear theory show that the value of the constant is zero for subsonic flow and four for supersonic flow about any symmetrical airfoil. These results are valid at Mach numbers appreciably less than or greater than unity but are invalid for Mach numbers near 1.

A similarity rule for the drag due to lift of a family of affinely related symmetrical airfoils that is valid throughout the Mach number range may be obtained from equation (91) by setting $\tilde{A} = \infty$.

$$\widetilde{\left(\frac{\Delta c_d}{\alpha^2}\right)} = \mathcal{D}_\Delta(\xi, \infty, \tilde{\alpha}) = d_\Delta(\xi_o, \tilde{\alpha}) \qquad (95)$$

At a Mach number of unity, equation (95), for the drag due to lift, reduces to the following:

$$\widetilde{\left(\frac{\Delta c_d}{\alpha^2}\right)}_{\xi_o=0} = d_\Delta(0, \tilde{\alpha}), \qquad \left(\frac{\Delta c_d}{\alpha^2}\right)_{M_o=1} = \frac{1}{[U_o k(t/c)]^{1/3}}d_\Delta\left(0, \frac{\alpha}{t/c}\right) \qquad (96)$$

Equation (96), together with hypothesis (d), indicates that, at sonic speed, the drag-rise ratio $\Delta c_d/\alpha^2$ of airfoils of a single family varies inversely as the cube root of the thickness ratio. For very large values of $\tilde{\alpha}$, the thickness ratio cannot have any effect on Δc_d; therefore, Δc_d is proportional to the five-thirds power of the angle of attack. If hypothesis (e) is accepted, the variation of $\widetilde{\Delta c_d/\alpha^2}$ with ξ_o at $M_o=1$ is zero, since

$$\left[\frac{d}{d\xi_o}\widetilde{\left(\frac{\Delta c_d}{\alpha^2}\right)}_{\tilde{\alpha}=0}\right]_{\xi_o=0} = \oint \frac{dZ}{dx}\left[\frac{d}{d\xi_o}\widetilde{\left(\frac{C_p}{\tilde{\alpha}^2}\right)}_{\tilde{\alpha}\to 0}\right]d\left(\frac{x}{c}\right)$$

$$= \oint \frac{dZ}{dx}\left(\frac{2}{\tilde{\alpha}^2}\right)_{\tilde{\alpha}\to 0}d\left(\frac{x}{c}\right) = 0 \qquad (97)$$

If k is taken to be $M_o{}^2(\gamma+1)/U_o$, $(dC_p/dM_o)_{M_o=1}$ is given by equation (56), and the variation of $\Delta c_d/\alpha^2$ with M_o at $M_o=1$ is

$$\left[\frac{d}{dM_o}\left(\frac{\Delta c_d}{\alpha^2}\right)_{\alpha\to0}\right]_{M_o=1}=\oint\frac{1}{\alpha}\frac{dZ}{dx}\left[\frac{d(C_p/\alpha)}{dM_o}\right]_{\alpha\to0}d\left(\frac{x}{c}\right)$$

$$=\oint\frac{1}{\alpha}\frac{dZ}{dx}\left[\frac{4}{\alpha(\gamma+1)}-\frac{2}{3}\left(\frac{C_p}{\alpha}\right)_{\alpha\to0}\right]d\left(\frac{x}{c}\right)=-\frac{2}{3}\left(\frac{\Delta c_d}{\alpha^2}\right)_{\alpha\to0} \quad (98)$$

The corresponding exact relation can be determined similarly by use of equation (52) for the slope of the pressure curve. It is

$$\left[\frac{d}{dM_o}\left(\frac{\Delta c_d}{\alpha^2}\right)_{\alpha\to0}\right]_{M_o=1}=-\frac{2}{\gamma+1}\left(\frac{\Delta c_d}{\alpha^2}\right)_{\substack{\alpha\to0\\M_o\to1}} \quad (99)$$

Wings of vanishing aspect ratio.—Since, as indicated in the preceding sections, the lift calculated by means of linear theory remains finite throughout the Mach number range, and diminishes to zero as the aspect ratio approaches zero, it follows that the drag due to lift must behave similarly. It is consequently reasonable to presume that linear theory is capable of describing the drag-due-to-lift characteristics of wings of vanishing aspect ratio at all Mach numbers. Therefore, the following relations stemming from equation (93) hold:

$$\left(\widetilde{\frac{\Delta C_D}{\alpha^2}}\right)_{\substack{l\\\tilde{A}\to0}}=\left(\widetilde{\frac{\Delta C_D}{\alpha^2}}\right)_{\substack{l\\\xi_o=0}}=\tilde{A}\times\text{const.},$$

$$\left(\frac{\Delta C_D}{\alpha^2}\right)_{\substack{l\\A\to0}}=\left(\frac{\Delta C_D}{\alpha^2}\right)_{\substack{l\\M_o=1}}=A\times\text{const.} \quad (100)$$

Solutions of the equations of linear theory show that the value of the constant is $\pi/4$ for all wings of small $\sqrt{|M_o^3-1|}A$ whose trailing edges possess no cutouts extending forward of the most forward station of maximum span (i. e., triangular, rectangular, elliptical wings, etc.). The quoted value of the constant corresponds to the development of the full "leading-edge force." It is known that this force is oftentimes not completely realized, due to a local separation and subsequent reattachment of the flow around the leading edge. If the leading-edge force is nonexistent, the corresponding value for the constant is $\pi/2$.

Wings of finite aspect ratio.—At the present time no drag-due-to-lift results have been obtained from the nonlinear theory for wings of either finite or infinite aspect ratio. The foregoing remarks, however, are sufficient to determine that a curve representing the variation of $\widetilde{\Delta C_D/\alpha^2}$ with $\tilde{A}$ for constant ξ_o and $\tilde{\alpha}$ would increase linearly with $\tilde{A}$ for small $\tilde{A}$ (unless the degree of attainment of the leading-edge force also depends on $\tilde{A}$) and become independent of $\tilde{A}$ for large $\tilde{A}$. The resulting curve would presumably have the same general appearance as that shown in figure 12 for $\widetilde{C_L/\alpha}$.

It may sometimes be desired to present drag-due-to-lift results in terms of $\Delta C_D/C_L^2$ or $\Delta C_D/\alpha C_L$ rather than $\Delta C_D/\alpha^2$. The similarity rules for these quantities can be quickly deduced from the foregoing results.

AMES AERONAUTICAL LABORATORY
  NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
    MOFFETT FIELD, CALIF., *Dec. 16, 1952*

APPENDIX A

RELATION BETWEEN U_o AND a^* STATEMENTS OF THE TRANSONIC-FLOW EQUATIONS

Equations (3), (7), (12), (13), (15), and (17) were presented in the text as being applicable to the study of transonic, as well as supersonic, flow about thin wings. These equations repeated below as equations (A1) through (A6), will be referred to as the U_o statement of the problem since the perturbation velocities are taken about the free-stream velocity U_o. The perturbation potential φ is defined by

$$\varphi = -U_o x + \Phi \tag{A1}$$

The differential equation is

$$(1-M_o^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = k\varphi_x\varphi_{xx} \tag{A2}$$

The shock relation is

$$(1-M_o^2)(\varphi_{x_1}-\varphi_{x_2})^2 + (\varphi_{y_1}-\varphi_{y_2})^2 + (\varphi_{z_1}-\varphi_{z_2})^2$$
$$= k\left(\frac{\varphi_{x_1}+\varphi_{x_2}}{2}\right)(\varphi_{x_1}-\varphi_{x_2})^2 \tag{A3}$$

The boundary conditions are
at $x = -\infty$

$$(\varphi_x)_o = (\varphi_y)_o = (\varphi_z)_o = 0 \tag{A4}$$

at the wing surface

$$(\varphi_z)_{z=0} = U_o\left(\frac{\partial Z}{\partial x}\right) \tag{A5}$$

The pressure coefficient is given by

$$C_p = -\frac{2}{U_o}\varphi_x \tag{A6}$$

In the usual a^* statement of the transonic-flow equations (eqs. (24) through (28) in the text), it is assumed that all velocities are only slightly different from the critical speed of sound a^*. The perturbation potential is defined by

$$\varphi' = -a^*x + \Phi \tag{A7}$$

The differential equation is

$$\varphi'_{yy} + \varphi'_{zz} = \frac{\gamma+1}{a^*}\varphi'_x\varphi'_{xx} \tag{A8}$$

The shock relation is

$$(\varphi'_{y_1}-\varphi'_{y_2})^2 + (\varphi'_{z_1}-\varphi'_{z_2})^2 = \frac{\gamma+1}{a^*}\left(\frac{\varphi'_{x_1}+\varphi'_{x_2}}{2}\right)(\varphi'_{x_1}-\varphi'_{x_2})^2 \tag{A9}$$

If the perturbation analysis is carried out in a completely consistent manner, the boundary conditions are:
at $x = -\infty$

$$(\varphi'_x)_o = (M_o^2-1)\frac{a^*}{\gamma+1}, \quad (\varphi'_y)_o = (\varphi'_z)_o = 0 \tag{A10}$$

at the wing surface

$$(\varphi'_z)_{z=0} = a^*\left(\frac{\partial Z}{\partial x}\right) \tag{A11}$$

and the pressure coefficient is given by

$$C_p = -\frac{2}{a^*}[\varphi'_x - (\varphi'_x)_o] \tag{A12}$$

The relation between the U_o and a^* statements can be determined directly in the following manner. The differential equations and shock relations for φ' and φ will be the same if

$$\frac{\gamma+1}{a^*}\frac{\partial\varphi'}{\partial x} = k\frac{\partial\varphi}{\partial x} - (1-M_o^2)$$

or if

$$\varphi' = k\frac{a^*}{\gamma+1}\varphi - \frac{a^*(1-M_o^2)}{\gamma+1}x \tag{A13}$$

The boundary conditions for φ' corresponding to those stated for φ in equations (A4) and (A5) are:
at $x = -\infty$

$$\left.\begin{array}{l}(\varphi'_x)_o = k\frac{a^*}{\gamma+1}(\varphi_x)_o - \frac{a^*(1-M_o^2)}{\gamma+1} = -\frac{a^*(1-M_o^2)}{\gamma+1} \\ (\varphi'_y)_o = k\frac{a^*}{\gamma+1}(\varphi_y)_o = 0, \quad (\varphi'_z)_o = k\frac{a^*}{\gamma+1}(\varphi_z)_o = 0\end{array}\right\} \tag{A14}$$

at the wing surface

$$(\varphi'_z)_{z=0} = k\frac{a^*}{\gamma+1}(\varphi_z)_{z=0} = kU_o\frac{a^*}{\gamma+1}\left(\frac{\partial Z}{\partial x}\right) \tag{A15}$$

Finally, the pressure coefficient is given by

$$C_p' = -\frac{2}{a^*}[\varphi'_x - (\varphi'_x)_o] = -\frac{2k}{\gamma+1}\varphi_x = \frac{kU_o}{\gamma+1}C_p \tag{A16}$$

Comparison of the above equations with those given previously for the completely consistent a^* analysis reveals their identity if, and only if, k is taken as $(\gamma+1)/U_o$, that is,

$$C_p' = C_p \quad \text{if} \quad k = \frac{\gamma+1}{U_o} \tag{A17}$$

As far as obtaining the values of C_p is concerned, therefore, the conventional a^* statement of the problem may be regarded as a transformation of the more general U_o statement with k being defined as in equation (A17) or (19). As is evident from equation (A13), however, the local velocities found in the a^* analysis are only correct when the free-stream Mach number is unity.

The foregoing discussion has demonstrated that the conventional a^* statement of transonic-flow problems yields the same values for C_p as the U_o statement, provided k in the latter set of equations is equated to $(\gamma+1)/U_o$. It is indicated in the text, however, that transonic results obtained by use of other expressions for k (and, in particular, $k=M_o{}^2(\gamma+1)/U_o$) are superior in many important aspects to those obtained using $k=(\gamma+1)/U_o$. It is consequently not without interest to determine a generalized form of the a^* equations which correspond to the U_o equations with unspecified k. The resulting equations for the perturbation potential and the shock relation are:

$$\varphi' = -a^*x + \Phi \tag{A18}$$

$$\varphi'_{yy} + \varphi'_{zz} = k'\varphi'_x\varphi'_{xx} \tag{A19}$$

$$(\varphi'_{y_1}-\varphi'_{y_2})^2 + (\varphi'_{z_1}-\varphi'_{z_2})^2 = k'\left(\frac{\varphi'_{x_1}+\varphi'_{x_2}}{2}\right)(\varphi'_{x_1}-\varphi'_{x_2})^2 \tag{A20}$$

The boundary conditions are:

at $x=-\infty$

$$(\varphi'_x)_o = \frac{M_o{}^2-1}{k'}, \quad (\varphi'_y)_o = (\varphi'_z)_o = 0 \tag{A21}$$

at the wing surface

$$(\varphi'_z)_{z=0} = a^*\left(\frac{\partial Z}{\partial x}\right) \tag{A22}$$

and the pressure coefficient is given by

$$C_p' = -\frac{2}{a^*}[\varphi'_x - (\varphi'_x)_o] \tag{A23}$$

It can be readily verified that these equations will produce the same values for the pressure coefficient as equations (A1) through (A6), provided k and k' are related as follows:

$$\frac{k'}{k} = \frac{U_o}{a^*} \tag{A24}$$

Thus, if it is desired to obtain values for C_p by use of equations (A18) through (A23) that are the same as those given by equations (A1) through (A6) with $k=M_o{}^2(\gamma+1)/U_o$, k' should be equated to the following:

$$k' = \frac{M_o{}^2(\gamma+1)}{a^*} \tag{A25}$$

APPENDIX B

DERIVATION OF SIMILARITY RULES

The basic equations of linear theory and of nonlinear theory of transonic flow may be summarized as follows. The differential equations are:

$$(1-M_o^2)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=\begin{cases}0 & \text{Linear} \quad \text{(B1)}\\[2mm] k\dfrac{\partial\varphi}{\partial x}\dfrac{\partial^2\varphi}{\partial x^2} & \text{Nonlinear} \quad \text{(B2)}\end{cases}$$

The approximate shock relation used in the treatment of transonic flow is

$$(1-M_o^2)(\varphi_{x_1}-\varphi_{x_2})^2+(\varphi_{y_1}-\varphi_{y_2})^2+(\varphi_{z_1}-\varphi_{z_2})^2$$

$$=k\left(\frac{\varphi_{x_1}+\varphi_{x_2}}{2}\right)(\varphi_{x_1}-\varphi_{x_2})^2 \quad \text{(B3)}$$

The boundary conditions are:
at $x=-\infty$

$$\left(\frac{\partial\varphi}{\partial x}\right)_o=\left(\frac{\partial\varphi}{\partial y}\right)_o=\left(\frac{\partial\varphi}{\partial z}\right)_o=0 \quad \text{(B4)}$$

at the wing surface

$$\frac{1}{U_o}\left(\frac{\partial\varphi}{\partial z}\right)_{z=0}=\tau\frac{\partial}{\partial(x/c)}f\left(\frac{x}{c},\frac{y}{b}\right) \quad \text{(B5)}$$

where the geometry of the wing is given by

$$(Z/c)=\tau f(x/c,\,y/b) \quad \text{(B6)}$$

The pressure coefficient is given by

$$C_p=-\frac{2}{U_o}\frac{\partial\varphi}{\partial x} \quad \text{(B7)}$$

If the differential equations are now transformed into a system with primed quantities and the proportionality or stretching factors are denoted by s with appropriate subscripts such that

$$\left.\begin{aligned}x'=s_x x,\quad y'=s_y y,\quad z'=s_z z,\quad \varphi'=s_\varphi\varphi,\quad U_o'=s_U U_o\\[2mm]\sqrt{(1-M_o^2)'}=s_\beta\sqrt{1-M_o^2},\quad U_o'k'=s_k U_o k \text{ or } k'=\frac{s_k}{s_U}k\end{aligned}\right\} \quad \text{(B8)}$$

the potential equations (eqs. (B1) and (B2)) become

$$\frac{s_x^2}{s_\beta^2 s_\varphi}(1-M_o^2)'\frac{\partial^2\varphi'}{\partial x'^2}+\frac{s_y^2}{s_\varphi}\frac{\partial^2\varphi'}{\partial y'^2}+\frac{s_z^2}{s_\varphi}\frac{\partial^2\varphi'}{\partial z'^2}$$

$$=\begin{cases}0 & \text{Linear} \quad \text{(B9)}\\[2mm]\dfrac{s_U s_x^3}{s_k s_\varphi^2}k'\dfrac{\partial\varphi'}{\partial x'}\dfrac{\partial^2\varphi'}{\partial x'^2} & \text{Nonlinear} \quad \text{(B10)}\end{cases}$$

and the shock relation becomes

$$\frac{s_x^2}{s_\beta^2 s_\varphi^2}(1-M_o^2)'(\varphi'_{x'_1}-\varphi'_{x'_2})^2+\frac{s_y^2}{s_\varphi^2}(\varphi'_{y'_1}-\varphi'_{y'_2})^2+$$

$$\frac{s_z^2}{s_\varphi^2}(\varphi'_{z'_1}-\varphi'_{z'_2})^2=\frac{s_U s_x^3}{s_k s_\varphi^3}k'\left(\frac{\varphi'_{x'_1}+\varphi'_{x'_2}}{2}\right)(\varphi'_{x'_1}-\varphi'_{x'_2})^2$$

$$\text{Nonlinear} \quad \text{(B11)}$$

The flow in the primed system is similar to that in the original system if φ' satisfies the same differential equations and boundary conditions as φ. Consequently, for similarity to exist, it is necessary first of all that the potential equations and the shock equations for the two flows be the same. Therefore, the following relations between the stretching factors must be satisfied:

$$\frac{s_y s_\beta}{s_x}=1,\quad \frac{s_y}{s_z}=1,\quad \frac{s_\varphi}{s_x s_U}\begin{cases}\lambda/\lambda' & \text{Linear} \quad \text{(B12)}\\[2mm]s_\beta^2/s_k & \text{Nonlinear} \quad \text{(B13)}\end{cases}$$

where, for linear theory, λ/λ' is an arbitrary constant which can be equated to s_β^2/s_k if desired. The constant is written as a fraction in order to maintain a certain symmetry throughout the analysis.

An immediate consequence of this transformation is that the wing plan forms undergo an affine transformation such that the aspect ratios of wings in similar flow fields are related, according to both linear and nonlinear theory, by

$$A'=\frac{s_y}{s_x}A=\frac{1}{s_\beta}A=\sqrt{\frac{1-M_o^2}{(1-M_o^2)'}}A \quad \text{(B14)}$$

or by

$$\sqrt{(1-M_o^2)'}\,A'=\sqrt{1-M_o^2}\,A \quad \text{(B15)}$$

Since φ' is proportional to φ, the boundary conditions at $x=-\infty$ are automatically satisfied. The boundary conditions at the wings may be given in either of two forms:

$$\left(\frac{\partial\varphi'}{\partial z'}\right)_{z'=0}=\frac{s_\varphi}{s_z}\left(\frac{\partial\varphi}{\partial z}\right)_{z=0}=\frac{s_\varphi}{s_z}U_o\tau\frac{\partial}{\partial(x/c)}f\left(\frac{x}{c},\frac{y}{b}\right) \quad \text{(B16)}$$

$$\left(\frac{\partial\varphi'}{\partial z'}\right)_{z'=0}=U_o'\tau'\frac{\partial}{\partial(x'/c')}f'\left(\frac{x'}{c'},\frac{y'}{b'}\right)=s_U U_o\tau'\frac{\partial}{\partial(x'/c')}f'\left(\frac{x'}{c'},\frac{y'}{b'}\right) \quad \text{(B17)}$$

whence, if the two wings have the same ordinate-distribution functions, that is, if $f'(x'/c',\,y'/b')=f(x/c,\,y/b)$, the ordinate-amplitude parameters are related as follows:

$$\tau'=\frac{s_\varphi}{s_U s_z}\tau=\frac{s_\varphi s_\beta}{s_U s_x}\tau$$

$$=\begin{cases}\dfrac{\lambda}{\lambda'}\sqrt{\dfrac{(1-M_o^2)'}{1-M_o^2}}\,\tau & \text{Linear} \quad \text{(B18)}\\[3mm]\dfrac{s_\beta^3}{s_k}\tau=\left[\dfrac{(1-M_o^2)'}{1-M_o^2}\right]^{3/2}\left(\dfrac{U_o k}{U_o'k'}\right)\tau & \text{Nonlinear} \quad \text{(B19)}\end{cases}$$

or as

$$\frac{\sqrt{(1-M_o^2)'}}{\lambda'\tau'}=\frac{\sqrt{1-M_o^2}}{\lambda\tau} \qquad \text{Linear} \qquad (B20)$$

$$\frac{\sqrt{(1-M_o^2)'}}{(U_o'k'\tau')^{1/3}}=\frac{\sqrt{1-M_o^2}}{(U_o k\tau)^{1/3}} \qquad \text{Nonlinear} \qquad (B21)$$

The relation between the pressure coefficients at corresponding points on the wing surface is given by

$$C_p'=-\frac{2}{U_o'}\left(\frac{\partial\varphi'}{\partial x'}\right)_{x'=0}=\frac{s_\varphi}{s_U s_x}\left(-\frac{2}{U_o}\frac{\partial\varphi}{\partial x}\right)_{x=0}=\frac{s_\varphi}{s_U s_x}C_p$$

$$=\begin{cases}\dfrac{\lambda}{\lambda'}C_p & \text{Linear} \qquad (B22)\\[2mm] \dfrac{s_\beta^2}{s_k}C_p=\left(\dfrac{\tau'}{\tau}\right)^{2/3}\left(\dfrac{U_o k}{U'_o k'}\right)^{1/3}C_p & \text{Nonlinear} \qquad (B23)\end{cases}$$

or more completely by

$$C_p'\left[\frac{\sqrt{(1-M_o^2)'}}{\lambda'\tau'}, \sqrt{(1-M_o^2)'}\;A';\frac{x'}{c'},\frac{y'}{b'}\right]$$

$$=\frac{\lambda}{\lambda'}C_p\left(\frac{\sqrt{1-M_o^2}}{\lambda\tau}, \sqrt{1-M_o^2}\;A;\frac{x}{c},\frac{y}{b}\right) \qquad \text{Linear} \qquad (B24)$$

$$C_p'\left[\frac{\sqrt{(1-M_o^2)'}}{(U_o'k'\tau')^{1/3}}, \sqrt{(1-M_o^2)'}\;A';\frac{x'}{c'},\frac{y'}{b'}\right]$$

$$=\left(\frac{U_o k}{U_o'k'}\right)^{1/3}\left(\frac{\tau'}{\tau}\right)^{2/3}C_p\left[\frac{\sqrt{1-M_o^2}}{(U_o k\tau)^{1/3}}, \sqrt{1-M_o^2}\;A;\frac{x}{c}\frac{y}{b}\right]. \qquad \text{Nonlinear} \qquad (B25)$$

The foregoing relationships may be summarized in the following statement: The similarity rule for the pressure coefficients on a family of wings having their geometry given by

$$(Z/c)=\tau f(x/c, y/b) \qquad (B26)$$

is

$$C_{p_l}=\frac{1}{\lambda}\mathcal{P}_l\left(\frac{\sqrt{1-M_o^2}}{\lambda\tau}, \sqrt{1-M_o^2}\;A;\frac{x}{c},\frac{y}{b}\right) \qquad \text{Linear} \qquad (B27)$$

$$C_p=\frac{\tau^{2/3}}{(U_o k)^{1/3}}\mathcal{P}\left[\frac{\sqrt{1-M_o^2}}{(U_o k\tau)^{1/3}}, \sqrt{1-M_o^2}\;A;\frac{x}{c},\frac{y}{b}\right] \qquad \text{Nonlinear} \qquad (B28)$$

The similarity rules for the lift, pitching-moment, and drag coefficients given by the linear theory are

$$C_{L_l}=\frac{1}{\lambda}\mathcal{L}_l\left(\frac{\sqrt{1-M_o^2}}{\lambda\tau}, \sqrt{1-M_o^2}\;A\right) \qquad \text{Linear} \qquad (B29)$$

$$C_{m_l}=\frac{1}{\lambda}\mathcal{M}_l\left(\frac{\sqrt{1-M_o^2}}{\lambda\tau}, \sqrt{1-M_o^2}\;A\right) \qquad \text{Linear} \qquad (B30)$$

$$C_{D_l}=\frac{\tau}{\lambda}\mathcal{D}_l\left(\frac{\sqrt{1-M_o^2}}{\lambda\tau}, \sqrt{1-M_o^2}\;A\right) \qquad \text{Linear} \qquad (B31)$$

The corresponding similarity rules given by nonlinear theory are

$$C_L=\frac{\tau^{2/3}}{(U_o k)^{1/3}}\mathcal{L}\left[\frac{\sqrt{1-M_o^2}}{(U_o k\tau)^{1/3}}, \sqrt{1-M_o^2}\;A\right] \qquad \text{Nonlinear} \qquad (B32)$$

$$C_m=\frac{\tau^{2/3}}{(U_o k)^{1/3}}\mathcal{M}\left[\frac{\sqrt{1-M_o^2}}{(U_o k\tau)^{1/3}}, \sqrt{1-M_o^2}\;A\right] \qquad \text{Nonlinear} \qquad (B33)$$

$$C_D=\frac{\tau^{5/3}}{(U_o k)^{1/3}}\mathcal{D}\left[\frac{\sqrt{1-M_o^2}}{(U_o k\tau)^{1/3}}, \sqrt{1-M_o^2}\;A\right] \qquad \text{Nonlinear} \qquad (B34)$$

It should be noted that the foregoing equations have been written for subsonic flow where $M_o \leq 1$. If $M_o \geq 1$, the radical $\sqrt{1-M_o^2}$ should be replaced with $\sqrt{M_o^2-1}$.

In the linear-theory analysis, λ has remained a completely arbitrary coefficient to be selected as best suits the particular problem at hand. For instance, the compressible-flow relationships between two wings having identical pressure distributions are found by setting $\lambda=1$. If, on the other hand, λ is set equal to $\sqrt{1-M_o^2}$, the thickness ratio, camber, and angle of attack of related wings are identical. The greatest simplification of the similarity rules of linear theory occurs when λ is set equal to $\sqrt{1-M_o^2}/\tau$ since the number of parameters necessary to show the results of linear theory is thereby decreased by one. This degree of arbitrariness in the similarity rules for linear theory is in contrast to the case for nonlinear transonic theory in which no undetermined coefficient like λ is to be found.

For the present purpose of gaining a better understanding of transonic flows, the most advantageous choice for λ is

$$\lambda=\frac{(U_o k)^{1/3}}{\tau^{2/3}} \qquad (B35)$$

because then the similarity rules for linear theory assume forms identical to those for nonlinear transonic theory. This is important since it implies that solutions of linear theory and of nonlinear transonic theory can be expressed as functions of the same parameters and, hence, can both be plotted on a single graph in terms of one set of parameters. The two theories would, of course, yield two distinct curves on such a plot. The curve for linear theory would be accepted as valid for purely subsonic and purely supersonic flows but may or may not be valid in the transonic range, as discussed previously. The curve for nonlinear transonic theory is valid not only for transonic flows, but for subsonic and supersonic small perturbation flows as well. As is evident from the derivation of the basic equations, however, the results of the nonlinear transonic theory should be considered to be of only equal accuracy to those of linear theory in the definitely subsonic and supersonic regimes, despite the fact that the solutions are much more difficult to obtain.

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